DOCUMENT RESUME

ED 159 076 SE 024 957

AUTHOR	Haag, V. H.; And Others
TITLE	Introduction to Secondary School Mathematics, Volume 1, Student's Text, Part I, Unit 37. Revised Edition.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	62
NOTE	194p.; For related documents, see SE 024 958-962
EDRS PRICE	MF-$0.83 HC-$10.03 Plus Postage.
DESCRIPTORS	Curriculum; Geometric Concepts; *Grade 7; *Instructional Materials; Junior High Schools; Mathematics Education; Number Concepts; Secondary Education; *Secondary School Mathematics; *Textbooks
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This is part one of a two-part SMSG text for grade seven students whose mathematical talents are underdeveloped. The reading level of this text has been adjusted downward, chapters shortened, and additional concrete examples included. Nevertheless, the authors warn that the text may not be appropriate for the very slow non-college-bound student. Chapter topics include: number symbols, whole numbers, non-metric geometry, and factoring and primes. (MN)

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Introduction to Secondary School Mathematics, Volume I

Student's Text, Part I

REVISED EDITION

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New Haven and London, Yale University Press

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Financial support for the School Mathematics
Study Group has been provided by the
National Science Foundation.

FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

CONTENTS

Chapter

1. WHAT IS MATHEMATICS? 1
    1-1. Mathematics as a Method of Reasoning. 1
    1-2. Mathematical Reasoning. 1
    1-3. From Arithmetic to Mathematics. 5
    1-4. Kinds of Mathematics. 7
    *1-5. Mathematical Discovery. 9
    *1-6. The Königsberg Bridge Problem 13
    1-7. Mathematics Today 14

2. NUMBER SYMBOLS 17
    2-1. Ancient Number Symbols. 17
    2-2. The Decimal System. 22
    2-3. Expanded Form and Exponents 27
    2-4. Numerals in Base Five 34
    2-5. Addition and Subtraction in Base Five 42
    2-6. Multiplication in Base Five 49
    *2-7. Division in Base Five 53
    2-8. Changing Decimal Numerals to Base Five Numerals. 55
    2-9. Numerals in Other Bases 57
    2-10. Summary 64
    2-11. Chapter Review. 66

3. WHOLE NUMBERS. 69
    3-1. Introduction. 69
    3-2. Sets. 69
    3-3. Counting Numbers. 73
    3-4. Properties of Operations. 75
    3-5. The Commutative Property. 75
    3-6. The Associative Property. 81
    3-7. The Distributive Property 86
    3-8. The Closure Property. 92
    3-9. Inverse Operations. 95
    3-10. Betweenness and the Number Line 100
    3-11. The Number One. 102
    3-12. The Number Zero 105
    3-13. Summary 109
    3-14. Chapter Review. 111
    3-15. Cumulative Review 112

4. NON-METRIC GEOMETRY I. 113
    4-1. Points, Lines, and Space. 113
    4-2. Planes. 120
    4-3. Names and Symbols 126
    4-4. Intersection of Sets. 133
    4-5. Intersections of Lines and Planes 137
    4-6. Summary 144
    4-7. Chapter Review. 147
    4-8. Cumulative Review 149

5. FACTORING AND PRIMES 153
- 5-1. The Building Blocks of Arithmetic 153
- 5-2. Multiples 154
- 5-3. Primes. 155
- 5-4. Factors 160
- 5-5. Divisibility. 166
- 5-6. Least Common Multiple 173
- 5-7. Summary 178
- 5-8. Chapter Review. 180
- 5-9. Cumulative Review 182
- Index . . . . .following page 183

PREFACE

To The Student:

This book will be your first adventure in secondary school mathematics. During this year you will develop a better understanding of what mathematics really is. We hope that you will enjoy this adventure.

Some of the big ideas in introductory high school mathematics are the following:

1. The numbers of arithmetic form a number system.

   You have studied whole numbers and fractions. You will see that these numbers are part of a larger system of numbers. You will also learn new ways to think about whole numbers and fractions.

2. Arithmetic leads to algebra.

   You will see how the ideas of algebra grow out of your knowledge of arithmetic.

3. Geometry helps us to understand the world in which we live.

   Ideas of points, lines, planes, and space are the alphabet of geometry. You will learn how these ideas form the basis for exploring the world of geometry.

This book was written for you. We hope that you will find it pleasant reading. There may be places where you will need the help of your teacher. As you read be sure to have a pencil and paper handy so that you can make computations and sketch diagrams.

The exercises are planned to give you practice in using the ideas in the text. Before trying to work an exercise read it carefully, more than once, if necessary. In each problem, be sure that you understand what you have to work with and what is required. Some exercises are marked with stars. These are harder than the others. Don't get discouraged if you cannot handle them immediately. The exercises marked "Brainbusters" are different from the others. Most of them are slightly "off beat" but they will give you a chance to use your imagination.

We cannot possibly tell you all about your first adventure in secondary school mathematics in a few paragraphs. We hope, however, that as you study mathematics this year you will gain a much better idea of what mathematics is. We also hope that you will make every effort to learn as much of it as you can. Good Reading!

Chapter 1

WHAT IS MATHEMATICS?

1-1. Mathematics as a Method of Reasoning.

> "On an airplane trip, I was talking with the man next to me. He asked me about the kind of work I do. I told him I was a mathematician. He asked, 'Don't you get tired of adding columns of figures all day long?' I had to explain to him that adding can best be done by a machine. '*My job is mainly logical reasoning*.'"

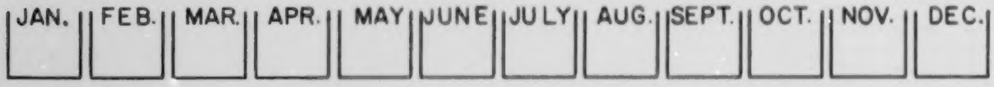
Just what is this mathematics that many people are talking about these days? Is it counting and computing? Is it drawing figures and measuring them? Is it a language which uses symbols like a mysterious code? No, mathematics is not any one of these. It is all of these and much more. Mathematics is a way of thinking and a way of reasoning.

1-2. Mathematical Reasoning.

Much of mathematics is concerned with "if-then" statements. By reasoning, mathematicians prove that *if* something is true, *then* something else must be true.

The following problem will help you to think like a mathematician. Suppose that there are 13 pupils in your class. Can you prove that at least two pupils in your class have birthdays in the same month?

Imagine 12 boxes, one for each month of the year. Imagine also that each pupil writes his birthday month on a slip of paper, and 12 of them place their slips in the correct boxes.



Can there be any empty boxes? Is it possible for any box to have two slips? Is it possible for any box to have more than two slips?

Is it possible that *every* box has less than two slips? How can this happen?

Suppose that each of the twelve pupils puts his slip in a different box. Then the 13th pupil puts his slip in the box where it belongs. Does any box have two slips?

By logical reasoning you have shown that if there are 13 pupils in the class, then at least two pupils have birthdays in the same month. You have proved an "if-then" statement. A mathematician's work is mainly to prove that if certain statements are true, then something else is true. He does this by logical reasoning.

"If-then" statements are not really strange to you. Here are a few:

If your birthday is on the 31st of a month, then you were not born in June.

If you arrive at school after the tardy bell, then you will be marked late to class.

If there are 60 students on a school bus and only 55 seats, then some students will have to stand up or wait for the next bus.

If you have 37 cents in your pocket, then you must have some pennies.

Can you think of some "if-then" statements?

Exercises 1-2a

(Class Discussion)

Use logical reasoning similar to the above problem to discuss the following exercises.

1. If you have a set of 5 pencils which you are going to distribute among 4 of your classmates, then some one of them will get at least 2 pencils.
2. If you gave 14 pencils to 6 classmates, then some one of them will get at least three pencils.

3. a. There are 7 movie houses in a town. What is the smallest number of people that would have to go to the movies to be sure that at least 2 persons see the same show?

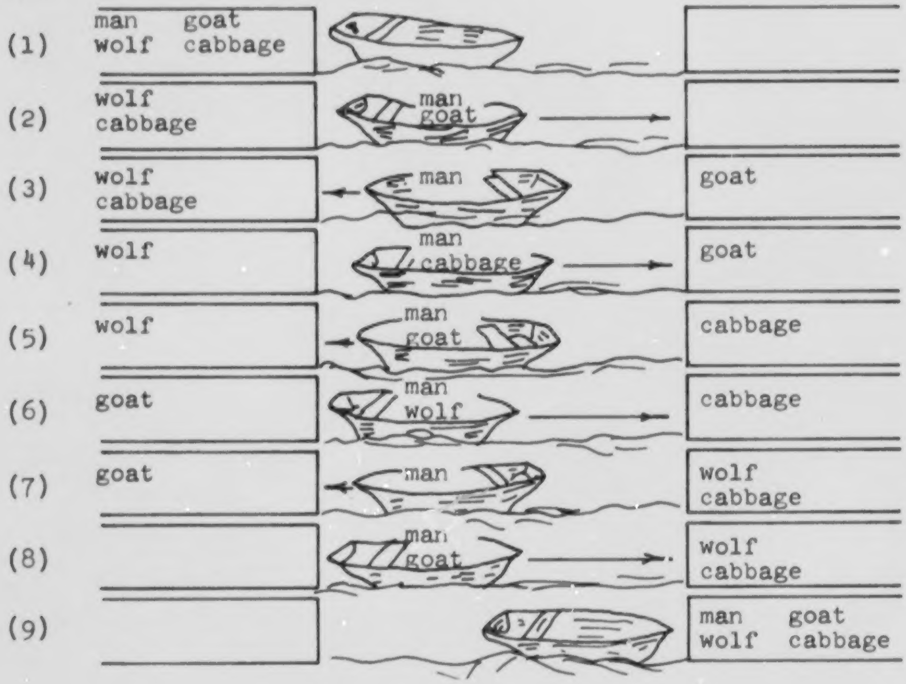
b. What is the smallest number of people that would have to go to the movies to be sure that at least 3 persons see the same show?

c. Reword parts a and b as "if-then" statements.

There are many interesting problems which need little arithmetic. Here is one in which you can arrive at the answer by reasoning.

> A man must carry a wolf, a goat, and a cabbage across a river in a boat so small that he can take only one at a time. He must always be on hand to keep the wolf from eating the goat, and the goat from eating the cabbage. In what way will he take them across the river?

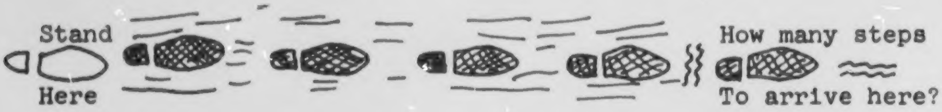
Solution:


(1)
man goat
wolf cabbage
(2)
wolf
cabbage
man
goat
(3)
wolf
cabbage
man
goat
(4)
wolf
man
cabbage
goat
(5)
wolf
man
goat
cabbage
(6)
goat
man
wolf
cabbage
(7)
goat
man
wolf
cabbage
(8)
man
goat
wolf
cabbage
(9)
man goat
wolf cabbage

Some of the following problems are very much like this one. All of these can be solved by reasoning. If you draw a picture it will help you discover the answer in many cases.

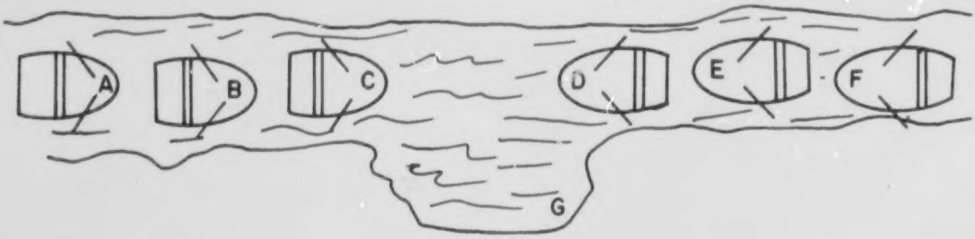
Exercises 1-2b

1. A little girl starts from a given point toward a wading pool. If each time that she takes two steps forward she must take one backward, how many steps will she have to take to step in the pool five steps ahead of her starting point?



2. How can two men and two small boys cross a river when the only boat is so small that it can carry at most one man or the two boys?

3. Boats A, B, C meet boats D, E, F in a river too narrow to allow them to pass. There is a small bay G which is large enough for any one of the boats. How can the boats pass each other? (They are allowed to row up and down the river as often as it is necessary.)

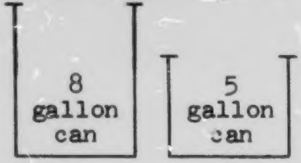


4. A 200-pound man and his two sons each weighing 100 pounds want to cross a river. If they have only one boat which can safely carry only 200 pounds, how can they cross the river?

5. A farmer wants to take a goose, a fox, and a bag of corn across a river. If left alone, the fox will eat the goose,

or the goose will eat the corn. If the farmer has only one boat large enough to carry him and one of the others, how does he cross the river?

6. Is it possible to measure out exactly 3 gallons using only an 8 gallon can and a 5 gallon can? The cans do not have individual gallon markings. Explain.



How can you measure out 3 gallons of water? (You may waste as much water as you please!)

7. Is it possible to measure out 2 gallons using the same cans as in Problem 6? Explain.

*8. Is it possible to measure out 1 gallon using the same cans as in Problem 6? Explain.

1-3. From Arithmetic to Mathematics.

The part of mathematics which you know best is arithmetic. You can often obtain results in arithmetic by experiment and by reasoning. This can save you a lot of hard work and time spent in calculation.

There is an interesting story told about a famous mathematician, Karl Friedrich Gauss. This happened many years ago, before 1800.

When Gauss was about 10 years old, his teacher told the children to add all the numbers from 1 to 100, that is

$$1 + 2 + 3 + \ldots + 100.$$

(Note: To save writing all the numbers between 3 and 100, it is customary to write three dots. This may be read "and so on.") In about two minutes Gauss stopped working. The teacher asked him why he wasn't working on the problem.

He replied, "I've done it already."

"Impossible!" exclaimed the teacher.

"It's easy," answered Gauss.

"First I wrote: $1 + 2 + 3 + 4 + \dots + 99 + 100,$

then I wrote the numbers in reverse order: $\underline{100} + \underline{99} + \underline{98} + \underline{97} + \dots + \underline{2} + \underline{1}.$

Then I added each column. $101 + 101 + 101 + 101 + \dots + 101 + 101$

When I added, I got one hundred 101's.

This gave me $100 \times 101 = 10{,}100$.

But I used each number twice; therefore my total is twice as large as it should be. So now I divided 10,100 by 2.

$$\frac{10{,}100}{2} = 5{,}050$$

In other words; $1 + 2 + 3 + \dots + 100 = 5{,}050$."

Exercises 1-3

1. Add all the numbers from 1 to 5, that is, $1 + 2 + 3 + 4 + 5$ using Gauss's method. Can you discover another short method different from the Gauss method?

2. Can Gauss's method be applied to the problem of adding the numbers: $0 + 2 + 4 + 6 + 8$?

3. By a short method add the odd numbers from 1 to 15, that is $1 + 3 + 5 + \dots + 15$. (To save writing all the numbers between 5 and 15 it is customary to write three dots. This may be read "and so on.")

4. This problem gives you a chance to make another discovery in mathematics.

	Add the numbers below:	Multiply the numbers below:
a.	$1 + 3 =$ ___?___.	$2 \times 2 =$ ___?___.
b.	$1 + 3 + 5 =$ ___?___.	$3 \times 3 =$ ___?___.
c.	$1 + 3 + 5 + 7 =$ ___?___.	$4 \times 4 =$ ___?___.

d. Look at the sums on the left and the products on the right. What seems to be the general rule for finding the sums of numbers on the left?

e. Apply your new rule to $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15$. Check with your answer in Exercise 3.

5. Add the odd numbers: $7 + 9 + 11 + 13 + 15 + 17$.

6. Add the even numbers: $4 + 6 + 8 + 10 + 12 + \ldots + 28$.

7. BRAINBUSTER. Add all the numbers from 1 to 200 by Gauss's method. Then add all the numbers from 0 to 200 by Gauss's method. Are the answers for these two problems the same? Why?

1-4. Kinds of Mathematics.

Mathematicians reason about all sorts of puzzling questions and problems as you can see by the above exercises. When they solve a problem they usually create a little more mathematics to add to the ever-growing store of mathematical knowledge. The new mathematics can then be used with the old to solve new problems. The total amount of mathematics is far greater than you can imagine. Arithmetic is one kind of mathematics.

Some of the other kinds of mathematics you may study in high school are algebra, geometry, and trigonometry. In algebra, we take a much closer look at addition, subtraction, multiplication, and division than is done in arithmetic. This leads to a new and very different mathematical language. In geometry you may study facts about points, lines, planes, and space. Trigonometry is necessary for further study in mathematics. Trigonometry is a branch of mathematics which grew out of the application of algebra to the study of triangles.

Today there are many different kinds of mathematics. No one mathematician can hope to learn more than a small bit of it. It would take the lifetime of a mathematician to be an expert in just one branch of mathematics. Hundreds of pages of new mathematics are being created every day of the year--much more than one person could possibly read in the same day. In fact, in the past 50 years more mathematics has been discovered than in all the thousands of years before that time.

Some Interesting Problems

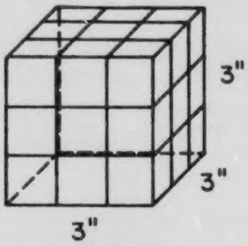
These problems are for fun and enjoyment, but they should begin to show you the general pattern of mathematics and may lead you to think as a mathematician thinks.

Exercises 1-4

1. Calvin Butterball sold his old used car for $90, bought it back for $80, and resold it for $100. If the original cost of the car had been $75, how much was Calvin's profit?

2. Without using pennies, see if you can figure out how many combinations of coins will make 30 cents.

3. Suppose you have a large wooden cube; that is, a block 3 inches wide, 3 inches long, and 3 inches high. This block is painted black.

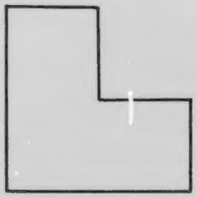


   a. How many cuts would you need to divide the cube into 1" cubes?
   b. How many cubes would you then have?
   c. How many cubes have 4 black sides?
   d. How many cubes have 3 black sides?
   e. How many cubes have 2 black sides?
   f. How many cubes have 1 black side?
   g. How many cubes are unpainted?

*4. Make all the numbers from 1 to 10 by using four 4's, never less and never more than four 4's. Example: $1 = \frac{44}{44}$; $2 = \frac{4}{4} + \frac{4}{4}$, $3 = \frac{4 + 4 + 4}{4}$

5. **BRAINBUSTER.** Two jealous husbands and their wives, Mr. and Mrs. Arnold and Mr. and Mrs. Bertrand, must cross a river in a boat that holds only two persons. How can this be done so that a wife is never left with the other woman's husband unless her own husband is present?

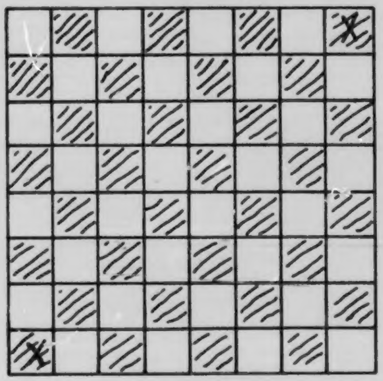
6. **BRAINBUSTER.** A man had a farm shaped like this: He had four sons and wanted to divide it so that each son would have a farm of exactly the same size and shape. How did he do it?



7. **BRAINBUSTER.** Suppose you have a checkerboard and dominoes. Each domino is just large enough to cover two squares on the checkerboard. Can you place the dominoes flat on the board in such a way as to cover all the board except two opposite corner squares?

Note: All the squares except the two squares in opposite corners are to be covered.

You may choose to leave the two white squares in opposite corners uncovered instead of the dark or shaded squares marked "x."



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*1-5. Mathematical Discovery.

Mathematics shows us how to deal with difficult problems. Mathematicians like to ask searching questions. Sometimes, hundreds of years pass before an answer is found. Sometimes, it is finally shown that there is no answer.

Many mathematical problems are still unsolved today. Also, many of the uses of mathematics have not yet been explored. For example, perhaps someday mathematicians will help to solve the highway traffic problem.

In this section, we will make a discovery about geometric figures. Which figures can be drawn without lifting your pencil from the paper or retracing your path? For example, let us consider the following figures:

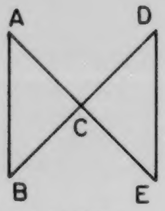


FIGURE 1

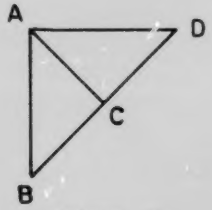


FIGURE 2

You should be able to copy these figures without lifting your pencil from the paper or retracing your path. (You may cross a path.) Try it in your notebooks.

Now, consider the following figures.

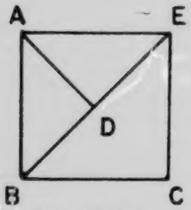


FIGURE 3

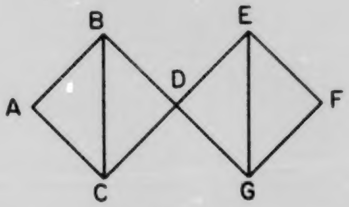


FIGURE 4

Can you copy these figures without lifting your pencil from your paper or retracing your path? You will find that you are unable to do it.

The following set of exercises will lead you to an interesting discovery about tracing figures.

Exercises 1-5

1. Of the nine figures shown, six can be drawn without lifting your pencil or retracing your path, while the others cannot. Which six can be drawn without lifting your pencil or retracing your path?

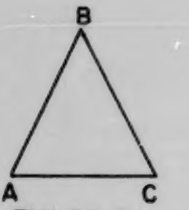


FIGURE 5

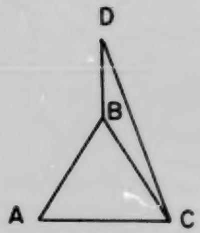


FIGURE 6

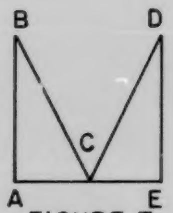
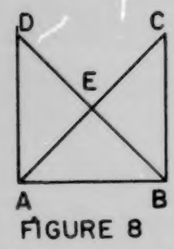


FIGURE 7



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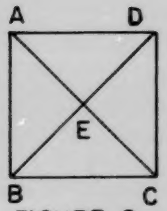

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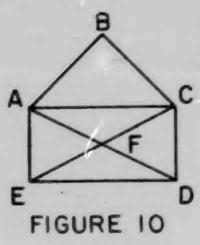


FIGURE 10

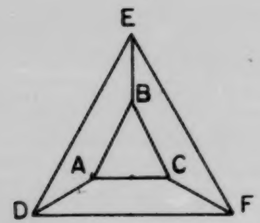


FIGURE 11

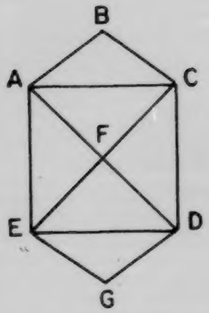


FIGURE 12

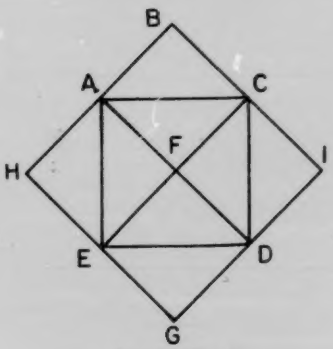


FIGURE 13

2. Count the number of paths which meet at each point. Copy the table in your notebook and complete it by placing your answer in the proper square. The first four figures are worked out for you.

FIGURE	A	B	C	D	E	F	G	H	I	NUMBER OF POINTS WITH ODD NUMBER OF PATHS	TRACEABLE? YES OR NO
1	2	2	4	2	2					0	Yes
2	3	2	3	2						2	Yes
3	3	3	2	3	3					4	No
4	2	3	3	4	3	2	3			4	No
5											
6											
7											
8											
9											
10											
11											
12											
13											

3. Use the results of the above chart to guess the answer to following question. When do you think that a figure is not traceable?

With a little careful thought you should be able to understand the following explanation. When an even number of paths come together at a point it is possible to leave each time you enter. When an odd number of paths come together at a point you must start or stop at that point. Therefore, if a figure is traceable there can be at most two points where an odd number of paths come together--a starting point and a stopping point.

*1-6. The Königsberg Bridge Problem.

For thousands of years people have enjoyed working various kinds of problems. A good example of this is a problem concerning the Königsberg Bridges. In the early 1700's the city of Königsberg, Germany was connected by seven bridges. Many people in the city at that time were interested in finding if it were possible to walk through the city so as to cross each bridge exactly once. After a few trials you may be convinced that the answer is "no", but it is not easy to prove it by trial and error since there are a large number of ways of taking such a walk. You may be interested in knowing that a mathematician in the year 1736 did prove that this could not be done.

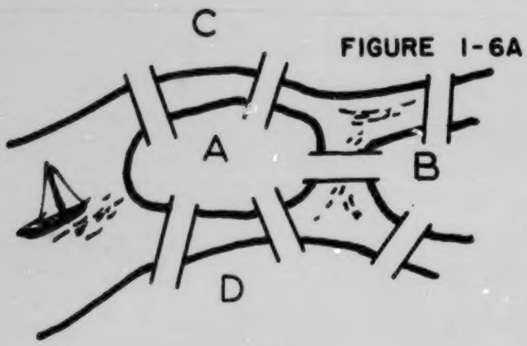


FIGURE 1-6A

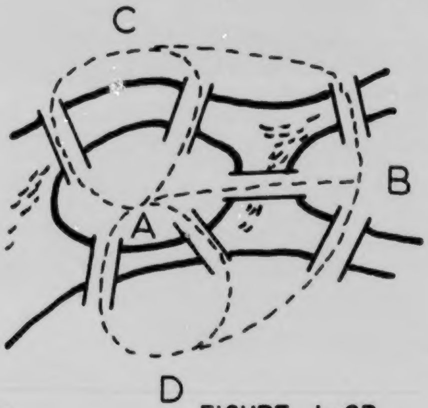


FIGURE 1-6B

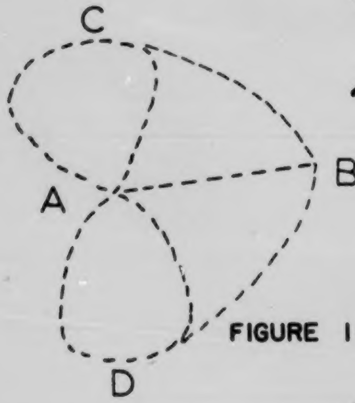


FIGURE 1-6C

Using the same method as we did in Exercises 1-5, let us look at this problem. Figure 1-6c shows how Figure 1-6a can be reduced to a line drawing. You know it will be impossible to pass just once over every bridge if there are more than two points where an odd number of routes meet.

Count the number of routes meeting at points A, B, C and D. Can you see why this problem is impossible?

1-7. Mathematics Today.

You are living in a world which is changing very rapidly. To get some idea of the changes in the past 20 years, ask your parents what life was like in their junior high school days. Did you realize that such things as color TV, nuclear ships, jet transports, satellites, space rocket-craft, and fusion power were all developed since you were born? There are new medicines and vaccines, new ways to make business decisions, new ways of computing and hundreds of other developments reported in the news every day. Even such things as the TV, telephone, radio, electric lights, airplane and the automobile were invented during the lifetimes of men living today.

Electronic computers are now solving, in a matter of minutes, problems which formerly would have taken months. Not only are mathematicians needed here but also many people with less training in mathematics. Unsolved problems are still waiting for faster computers to be made.

In the aircraft industry, mathematics is vital in designing the best shape of aircraft as well as designing radio and radar devices.

If you have ever visited a large telephone exchange building, you are aware of the complicated array of electronic equipment.

In the petroleum industry mathematics is used extensively in deciding how many oil wells to sink and where to drill to get the most oil from an oil field at the least cost. Mathematics and mathematicians have had a part in all of these areas and in many others.

For the first time, mathematics is now widely used and required in fields such as social studies, medical science, psychology, geology, and business. Mathematical reasoning skills are useful in all these fields. Also, much of the use of electronic computers in business and in industry is carried on by people who are not well trained in mathematics. These people usually must continue to learn more about mathematics and computing in order to develop mastery of their jobs. Merely to understand these phases of modern life, and to appreciate them enough to be a good citizen, you will need to know something about mathematics.

You may think that you will not need mathematics, but can you picture a day without mathematics? This is one of the major reasons why this textbook is being written. We know that you will need far more mathematics in your lifetime than your parents or your teachers were required to learn. We hope to be able to give you a better foundation for all your future studies in mathematics and in other sciences. At the same time we hope you will share in some of the fun and excitement that people enjoy in discovering and in using mathematics.

Chapter 2

NUMBER SYMBOLS

2-1. Ancient Number Symbols.

Since very early times, people have needed to use numbers. They used numbers for counting arrows, or sheep, or animal skins. The records of their numbers were very different from those we now use. When a boy counted sheep, he cut marks on a stick, or used a pile of pebbles. One mark, or one pebble stood for one sheep. He could tell later that all his sheep were present if he still had one sheep for each mark, or for each pebble. You make the same kind of record when you count votes in a class election. You make one mark for each vote, like this: 卌 ||. When people began to make marks for numbers they were writing the first numerals. Numerals are the symbols (or marks) made for numbers. Thus, the numeral "7" is the mark made to stand for the number seven. In this chapter we shall study some of the ways in which number symbols can be written.

In addition to marks, people also use sounds, or names, for numbers. Today a boy counting sheep uses the name "one" for a single sheep, "two" for 2 sheep, and so on. We now use both numerals (1, 2, 3, etc.) and words (one, two, three, etc.) to stand for numbers.

Egyptian Numerals

One of the earliest systems of writing numerals was used by the Egyptians about 5,000 years ago. They used a system of picture numerals with which they could write numbers up to millions. Egyptian symbols are shown below:

Our Number	Egyptian Symbol	Object Pictured
1	|	stroke or staff
10	∩	heel bone
100	𓍢	coiled rope or scroll
1,000	𓆼	lotus flower

Our Number	Egyptian Symbol	Object Pictured
10,000	𓂭	pointing finger
100,000	𓆐	polliwog
1,000,000	𓁨	astonished man

These symbols were carved on wood or stone. The Egyptian system was an improvement over the very earliest systems because it used these ideas:

1. A single symbol could be used to show the number of objects in a collection. For example, the heel bone stood for the number ten.
2. Symbols were repeated to show larger numbers. The group of symbols 𓍢𓍢𓍢 meant 100 + 100 + 100, or 300.
3. This system was based on groups of ten. Ten strokes made a heelbone, ten heelbones made a scroll, and so on.

The following table shows how Egyptians wrote numerals:

Our numerals	4	11	23	20,200	1959
Egyptian numerals	||||	∩|	∩∩|||	𓂭𓂭𓍢𓍢	𓁨𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢∩∩∩∩∩|||||||||

Exercises 2-1a

1. Show how the following would be written with Egyptian numerals:

a. 3	c. 30	e. 1960
b. 12	d. 225	f. 2,300,000

2. What number is shown by each of these?

a. ∩∩∩||	c. 𓍢𓍢𓍢∩∩∩∩
b. 𓍢𓍢∩||||	d. 𓁨𓍢𓍢∩∩∩∩∩

3. The Egyptians usually followed a pattern in writing large numbers. However, the meaning was not changed if the symbols were written in a different order. For example, ∩∩| and ∩|∩ both meant twenty-one.
Write thirty-two in as many ways as you can, using Egyptian symbols.

4. Use two different kinds of numerals to write each of these:
   a. seven   b. fifteen   c. two hundred four
   d. ten thousand, three hundred, fifty-one.

Babylonian Numerals

About 4,000 years ago, the Babylonians developed a very different way of writing numerals. They had no paper, but did their writing with a piece of wood on a clay tablet. The pieces of wood were wedge-shaped at the ends. By pressing one end of the wedge into the clay, a mark like this 𒁹 could be made, to mean "one." The other end made a mark 𒌋 which meant "ten." When the clay tablets dried they became very hard. Many of these tablets have been found in recent years. By studying them, mathematicians can learn much about the Babylonian system of numerals.

Babylonian symbols were used to write numerals up to 59 by repeating symbols just as the Egyptians did. Some of their numerals are shown in the table.

Our numerals	5	13	32	59
Babylonian numerals	𒁹𒁹𒁹 𒁹𒁹	𒌋𒁹𒁹𒁹	𒌋𒌋𒌋𒁹𒁹	𒌋𒌋 𒁹𒁹𒁹 𒌋𒌋 𒁹𒁹𒁹 𒌋 𒁹𒁹𒁹

Although the Babylonians used these same symbols in different positions to write numbers larger than 59, we find their method inconvenient.

Roman Numerals

The Roman system was used by Romans and many other peoples for hundreds of years. We still use it sometimes, though it is not the best system for us.

Historians believe that the Roman numerals came from pictures of fingers, like this: |, ||, |||, ||||. The Romans then used a hand for five, . Gradually some of the marks were omitted, and they wrote V for five. Two fives put together made the symbol for ten, X. The following table shows symbols used by the Romans.

Our Numeral	1	5	10	50	100	500	1000
Roman Numeral	I	V	X	L	C	D	M

To write larger numbers, symbols were repeated, or added. For example:

XX = 10 + 10 or 20,   XVI = 10 + 5 + 1 or 16, and
DCVII = 500 + 100 + 5 + 1 + 1 or 607.

Later the Romans used the subtraction idea to write certain numbers. Thus, IV means 5 - 1 or 4, and IX means 10 - 1 or 9. The subtraction idea was used only for showing numbers related to four and nine as follows:

IV = 4   XL = 40   CD = 400
IX = 9   XC = 90   CM = 900

In the Roman system, the position of a symbol was important. Except for the examples shown for subtraction, the symbols for small numbers always followed symbols for larger numbers. Thus, XL meant 40, while LX meant 60.

The Roman symbols could not be used with any ease in computation. Instead, the Romans used a form of counting board. The example below will show how hard multiplication with Roman numerals could be. The example shows 235 × 2, which for us is a very easy problem.

Multiply:

$$\begin{array}{r} \text{CCXXXV} \\ \text{II} \\ \hline \text{CCCCXXXXXXVV} \end{array} \quad \text{or} \quad \text{CDLXX}$$

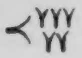
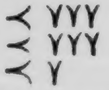
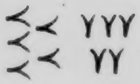
When you have some spare time, try multiplying 276 by 13 with Roman symbols!

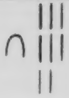
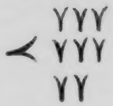
Many other systems of writing numerals have been used; the Korean, Chinese, Japanese, and Indian systems in Asia; the Mayan, Incan, and Aztec systems of the Americas; the Hebrew, Greek, and Arabian systems in the Mediterranean regions. You would find the study of these systems very interesting, but we cannot discuss all of them in this chapter.

Decimal Numerals

The way we write numerals was developed by the Hindus in India. The Arabs, who were great travelers, learned it and carried it to Europe. For this reason it is sometimes called the Hindu-Arabic system. The symbols have changed very much, however, from those used by the Hindus or the Arabs. Our system is called a "decimal" system because it is built on groups of ten. The word "decimal" comes from a Latin word meaning ten.

Exercises 2-1b

1. Write the following in Babylonian numerals:
   a. 4 b. 11 c. 23 d. 38
2. Write the decimal numerals for each of these numbers:
   a. 
   b. 
   c. 
3. Write the decimal numerals for each of these:
   a. XVI
   b. XIV
   c. XXIX
   d. CX
   e. XC
   f. CV
   g. DCLXVI
   h. MMCCCL
4. Write these as Roman numerals:
   a. 15 c. 34 e. 98 g. 3,256
   b. 23 d. 62 f. 629
5. Make a table like the one below to show different ways in which number symbols might be written.

Number	Decimal System	Roman	Egyptian	Babylonian
Sample: Eighteen a. Six b. Seventeen c. Twenty-four	18	XVIII		

6. How many different symbols were used to write numerals in the
   - a. Egyptian system?
   - b. Babylonian system?
   - c. Roman system?
   - d. Decimal system?

2-2. The Decimal System.

The decimal system has two important advantages over the older systems.

1. Ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 may be used to express any number, as large or as small as we wish. There is no need to invent a new symbol for a very large number, as the Romans or Egyptians had to do.
2. Computation (subtracting, multiplying, and so on) can be performed very easily.

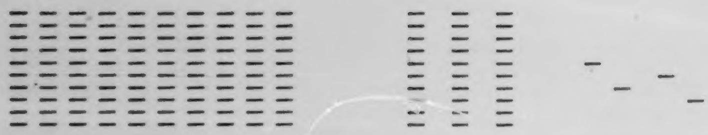
These two advantages of the decimal system come from its use of zero and place value. Place value will be explained in a later section.

The decimal system is now used in most parts of the world. It is like an international language for numbers. Since its symbols and the positions in which they are written have very precise meanings, we must be careful to use them correctly.

Digits

The ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 used in our system are called digits. Each digit is the name for a number. A numeral may be written with one or several digits. For example, the numeral 718 contains three digits; 2655 contains four digits.

If you had a great many pennies to count, you could arrange them first in stacks of ten pennies each. Then you might group the stacks to have ten stacks in each group. After this, it is very easy to state the total number of pennies. The diagram on the following page shows pennies arranged in this way. Before reading any farther, find as quickly as you can how many pennies are shown. Notice how much the counting is helped by having the pennies grouped.



The idea of grouping is one of the important ideas of the decimal system. Objects are grouped by tens. Ten groups of ten make a larger group called one hundred. Ten groups of one hundred make one thousand, and so on. We say that ten is the base of the system.

Place Value

The numeral for the number of pennies above is 134. When we write this we show the four single pennies by a "4" in the ones place as in the table below. The three stacks of 10 pennies are shown by a "3" in the tens place. The one group of one hundred pennies is shown by a "1" in the hundreds place.

Number of Thousands	Number of Hundreds	Number of Tens	Number of Ones
	1	3	4

Do you see that the place or position of a digit in a numeral tells the size of the group that the digit counts? Do you see that each place has a value ten times as large as the place just to the right of it?

Each place in a decimal numeral has a name. The first place is the "one" place, the second the "ten" place, the third the "hundred" place, and so on. To simplify writing and reading numerals we use commas to separate groups of three digits, beginning at the right. Each group of three digits is given a name also. The table on the following page shows place names and group names for the first four groups.

Group Name	Billions	Millions	Thousands	Units
Place Name	Hundred Billion Ten Billion Billion	Hundred Million Ten Million Million	Hundred Thousand Ten Thousand Thousand	Hundred Ten One
Digits	5 4 5,	4 6 5,	7 3 8,	9 2 1

We read the numeral shown in the table as follows:

Numeral	Read
545,	five hundred forty-five billion
465,	four hundred sixty-five million
738,	seven hundred thirty-eight thousand
921	nine hundred twenty-one

The whole numeral shown in the table is correctly read as "five hundred forty-five billion, four hundred sixty-five million, seven hundred thirty-eight thousand, nine hundred twenty-one." The word "and" is not used in reading numerals for whole numbers. In the same way, 23,207,325 is read: "twenty-three million, two hundred seven thousand, three hundred twenty-five."

Exercises 2-2a

1. Practice reading the following numerals orally. Use the table for help if you need it. We do not use the word "and" in reading whole numbers.
   a. 300
   b. 3,005
   c. 7,109
   d. 15,015

e. 234,000
f. 608,014
g. 100,009
h. 1,024,305
i. 30,250,089
j. 52,360,215,723

2. Write the digits used in the decimal system.

3. Starting with the ones place and counting from right to left, what is the name of the fourth place?

How many digits are there in these numerals?

a. 3184
b. 26
c. 9
d. 111
e. 24,307
f. 32,000.

5. a. Write four 3-digit numerals.
   b. Write a 5-digit numeral with a digit 0 in the hundred place.

6. a. Write the six-digit numeral with the smallest possible value.
   b. Write the numeral in words.
   c. Write a 4-digit numeral with the digit 9 in the tens place.

7. a. Write the six-digit numeral with the largest possible value.
   b. Write the numeral in words.

8. Write the following in numerals:
   a. one hundred fifty-nine
   b. five hundred two
   c. five thousand, two hundred
   d. six thousand, eight hundred fifty-seven
   e. twenty-seven thousand, seventeen
   f. one hundred eleven thousand
   g. three million, three thousand, three
   h. five billion, two
   i. the number following two thousand ninety-nine
   j. the number following five thousand, nine hundred ninety-nine.

9. Name the larger of the two numbers:
   a. 857 or 785
   b. 333,000 or 330,300
   c. 909 or 910
   d. 330,000 or 33,000

10. How many times as large as the first number is the second?

a. 7, 70

b. 70, 700

c. 230, 2300

d. 654, 6540

e. 8, 800

f. 37, 37,000

g. 2040, 20,400

h. 5600, 5,600,000

11. Ten billion is how many times as large as one hundred million?

Exercises 2-2b

Here are some review exercises in using the numerals of the decimal system in the four operations of arithmetic. Try to do them without making any errors.

Add:

1. $$\begin{array}{r} 24\\ 35\\ 20\\ \underline{56} \end{array}$$

2. $$\begin{array}{r} 87\\ 64\\ 76\\ \underline{27} \end{array}$$

3. $$\begin{array}{r} 309\\ 824\\ 630\\ \underline{95} \end{array}$$

Subtract:

4. $$\begin{array}{r} 69\\ \underline{34} \end{array}$$

5. $$\begin{array}{r} 375\\ \underline{97} \end{array}$$

6. $$\begin{array}{r} 813\\ \underline{208} \end{array}$$

Multiply:

7. $$\begin{array}{r} 564\\ \underline{7} \end{array}$$

8. $$\begin{array}{r} 407\\ \underline{38} \end{array}$$

9. $$\begin{array}{r} 5140\\ \underline{65} \end{array}$$

10. $$\begin{array}{r} 3900\\ \underline{840} \end{array}$$

Divide:

11. $4\overline{)496}$

12. $6\overline{)642}$

13. $53\overline{)31800}$

14. $28\overline{)8516}$

15. $125\overline{)503758}$

2-3. Expanded Form and Exponents.

Let us take some numerals apart to see exactly what they mean. Look at the three written below:

A. 5 B. 50 C. 500

In numeral A the "5" stands for 5 ones or, 5 × 1 because 5 is in the one place.

In numeral B the "5" stands for 5 tens or, 5 × 10 because 5 is in the ten place.

In numeral C the "5" stands for 5 hundreds or, 5 × 100 or, 5 × 10 × 10 because 5 is in the hundred place.

Each one of the numerals uses the digit 5. The meaning of the numeral changes as we write the "5" in a different place in the numeral. The value of the place where "5" is written in B is ten times as large as it is in A. In numeral C, the value of the place of "5" is ten times as large as it is in B. The same is true for all places in our numeral system. Each place has a value ten times as large as the place to the right of it.

Now let's look at the meanings of some other decimal numerals.

3 means 3 × 1
30 means 3 × 10
37 means (3 × 10) + (7 × 1)
26 means (2 × 10) + (6 × 1)
425 means (4 × 100) + (2 × 10) + (5 × 1)
or (4 × 10 × 10) + (2 × 10) + (5 × 1)

In the numeral 604, the "0" in the ten place shows that there are no tens, so 604 means (6 × 100) + (0 × 10) + (4 × 1). When numerals are written as above to show their meaning, we say they are written in "expanded form." In expanded form, 2345 = (2 × 1000) + (3 × 100) + (4 × 10) + (5 × 1).

The parentheses are used here to show that the whole expression inside is to be thought of as one numeral.

Exercises 2-3a

1. In which of the numerals below does "3" stand for 3 tens?
   2380; 432; 234; 653; 300; 38; 60,385.
2. In which of the numerals in Question 1 does the "3" stand for 3 hundreds?
3. Select the numerals in which "4" has the same place value.
   3456; 84; 143; 402; 4150; 56,420.
4. Write the following numerals in expanded form:
   a. 28
   b. 56
   c. 721
   d. 1312
   e. 244
   f. 2846
   g. 507
   h. 23,162

We say that the decimal system of writing numerals has place value and a base ten. Starting at the one place, the value of each place is multiplied by ten to get the value of the next place.

The table below shows some of the place values of a digit in a decimal numeral. Notice how the multiplier "ten" is used over and over.

Place value	Thousand	Hundred	Ten	One
Meaning	1000	100	10	1
	10 × 100	10 × 10	10 × 1	
	10 × 10 × 10			

When we write 2356 in expanded form we write:

2356 = (2 × 1000) + (3 × 100) + (5 × 10) + (6 × 1)
or (2 × 10 × 10 × 10) + (3 × 10 × 10) + (5 × 10) + (6 × 1)

The second way shows more clearly how the numeral is put together.

Oral Exercises 2-3b

Multiply:

1. 304 × 10
2. 304 × 1000
3. 275 × 100
4. 222 × 100
5. 20 × 30
6. 100 × 990
7. 200 × 70
8. 50 × 900
9. 600 × 80
10. 80 × 800

Divide:

11. 2700 ÷ 10
12. 2700 ÷ 100
13. 270 ÷ 10
14. 3050 ÷ 10
15. 10,000 ÷ 100
16. 9900 ÷ 30
17. 9900 ÷ 900
18. one million ÷ one thousand
19. one thousand ÷ one hundred
20. one hundred ÷ one hundred

Exercises 2-3c

Write the following numerals in expanded form, thus:

$$335 = (3 \times 10 \times 10) + (3 \times 10) + (5 \times 1).$$

1. 423
2. 771
3. 5253
4. 2608

*5. 34,359

In writing the preceding exercises, you may have found it tiresome to write so many 10's. Is there a shorter way to write an expression such as 10 × 10 × 10 × 10?

Several years ago you learned that there is a short way to write 10 + 10 + 10 + 10. What is it?

We know that 10 + 10 + 10 + 10 can be written "4 × 10," since each expression is equal to forty. Similarly we can write:

2 + 2 + 2 + 2 + 2 as 5 × 2

and 2 + 2 + 2 as 3 × 2.

In the last example, the "3" shows three 2's in the addition problem.

Now let's look at: 2 × 2 × 2

Again we have three 2's, but this time they are multiplied together. Can we write a "3" somewhere to indicate this case?

How about 3 × 2? No, this means 6, and 2 × 2 × 2 = 8

How about 32? No, this means (3 × 10) + (2 × 1) or thirty-two.

Mathematicians have agreed that they will write "3" and "2" to show 2 × 2 × 2 in this way: 2^3. Our new symbol 2^3 is read "2 to the third power."

Then 2 × 2 × 2 × 2 × 2 could be written as 2^5 and 10 × 10 × 10 × 10 as 10^4.

Exercises 2-3d

1. Write these to show their meaning. Thus:
   $2^5 = 2 \times 2 \times 2 \times 2 \times 2$
   - a. 4^3
   - b. 3^4
   - c. 5^2
   - d. 2^5
   - e. 2^3
   - f. 3^2
   - g. 4^5
   - h. 5^4
2. Find the value of each of these expressions. Use your answers from Exercise 1 to help you.
   - a. 4×3
   - b. 4^3
   - c. 3^4
   - d. 5×2
   - e. 5^2
   - f. 2^5
   - g. 2×3
   - h. 2^3
   - i. 3^2
   - j. 4×5
   - k. 5×4
   - l. 4^5
   - m. 5^4
3. Write these expressions in words.
   Example: 5^2 is read "five to the second power."
   - a. 5^3
   - b. 10^6
   - c. 2^5
   - d. 10^4
   - e. 2^3
   - f. 8^2
4. Write each of these expressions in shorter form like this:
   (1) $3 + 3 + 3 + 3 = 4 \times 3$
   (2) $3 \times 3 \times 3 \times 3 = 3^4$
   - a. $2 + 2 + 2 + 2$
   - b. $10 + 10 + 10$
   - c. $10 + 10 + 10 + 10$
   - d. $6 + 6 + 6 + 6 + 6$
   - e. $8 + 8 + 8$
   - f. $2 \times 2 \times 2 \times 2$
   - g. $10 \times 10 \times 10$
   - h. $10 \times 10 \times 10 \times 10$
   - i. $6 \times 6 \times 6 \times 6 \times 6$
   - j. $8 \times 8 \times 8$
   - k. $3 + 3 + 3 + 3 + 3$
   - l. $3 \times 3 \times 3 \times 3 \times 3$
   - m. 4×4
   - n. $4 + 4$
   - o. $5 \times 5 \times 5$
   - p. $2 \times 2 \times 2$
   - q. $2 + 2 + 2$

5. Write each of the following in a shorter manner.

   a. ten to the second power
   b. ten times two
   c. five to the fourth power
   d. two to the third power
   e. two times three
   f. three to the tenth power
   g. one to the tenth power
   h. ten to the first power

Exponent

When we write $10 \times 10 \times 10 \times 10$ as 10^4, the "4" is called the exponent. The "10" is called the base. When we write 5^3, the exponent is three, and the base is five.

An exponent always tells how many times the base is used as a factor. The factors are to be multiplied. For example:

$$5^3 \text{ means } 5 \times 5 \times 5.$$

The table on the following page shows how some of the important numerals of the decimal system are written with exponents.

Numeral	Meaning	Written as	Read as
100,000	$10 \times 10 \times 10 \times 10 \times 10$	10^5	ten to the fifth power
10,000	$10 \times 10 \times 10 \times 10$	10^4	ten to the fourth power
1,000	$10 \times 10 \times 10$	10^3	ten to the third power
100	10×10	10^2	ten to the second power
10	10	10^1	ten to the first power

Usually 10^1 is written simply as 10, without the exponent "1." All other exponents must be written. All the numbers 10^4, 10^3, 10^2, etc. are called powers of ten.

We may use exponents in writing numerals in expanded form. Instead of writing

$$352 = (3 \times 10 \times 10) + (5 \times 10) + (2 \times 1)$$

we may write $352 = (3 \times 10^2) + (5 \times 10) + (2 \times 1)$.

Instead of writing

$$6702 = (6 \times 10 \times 10 \times 10) + (7 \times 10 \times 10) + (0 \times 10) + (2 \times 1)$$

we may write $6702 = (6 \times 10^3) + (7 \times 10^2) + (0 \times 10) + (2 \times 1)$.

Exercises 2-3e

1. Write each of the following using exponents.
   a. $3 \times 3 \times 3 \times 3 \times 3$
   b. $6 \times 6 \times 6 \times 6 \times 6 \times 6$
   c. $25 \times 25 \times 25$
   d. $5 \times 5 \times 5 \times 5 \times 5 \times 5$
   e. $279 \times 279 \times 279 \times 279 \times 279$
   f. 16
2. How many fives are multiplied in each of the following?
   a. 5^3
   b. 5^7
   c. 5^2
   d. 5^{10}
   e. 5^5
3. Write each of the following without exponents as $2^3 = 2 \times 2 \times 2$.
   a. 2^8
   b. 10^7
   c. 33^5
   d. 60^6

4. Write each of the following expressions without exponents to show the meaning of the exponent. Then find the value of each expression as shown in the example:
4^3 means $4 \times 4 \times 4$ or 64.

a. 4^4
b. 6^2
c. 7^3
d. 8^2
e. 9^2
f. 10^3
g. 2^6
h. 4^5

5. Which numeral represents the larger number? Explain your answer.
a. 2^3 or 3^2
b. 4^3 or 3^4

6. Write the following numerals in expanded form as shown in the example: $210 = (2 \times 10^2) + (1 \times 10) + (0 \times 1)$.
a. 468
b. 5324
c. 7062
d. 59,126
e. 109,180

7. Copy and complete the following table:

	Powers of Ten	
Power	Numeral	Words
10^1	10	Ten
10^2		
10^3		
10^4		
10^5		
10^6		

8. Write the following numerals by using 10 with an exponent.
a. 1000
b. 100,000
c. 1,000,000
d. one hundred million

*9. From the table of powers of ten you completed in Question 7, how does the exponent for the base "ten" compare with the number of zeros when the numeral is written in the usual way?

10. BRAINBUSTER. What do you think 10^0 should mean? (Hint: What is the meaning of 10^2? of 10^1?)

11. **BRAINBUSTER.** A mathematician was talking to a group of kindergarten children one day. They were talking about, and writing, very large numbers. One child wrote 1 followed by 100 zeros. The name "googol" was later suggested as a name for this number. Write a googol using 10 and an exponent.

2-4. Numerals in Base Five.

It is natural that our numeral system should use ten as a base, since we have ten fingers. Children now often learn to count on their fingers. We should not think, however, that ten must be used as a base. The Celts, who lived in Europe more than 2000 years ago, used twenty as a base, and so did the Mayans in Central America. Can you think of a reason for this? Some Eskimo tribes group and count by fives. Can you see why this might be done? Some people think ten may not be the best base for us to use.

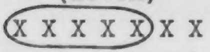
Suppose that in our travels through space we landed on planet Quintus and found people with only one hand. If the hand had five fingers, such people might count by fives instead of tens. Let's see how a numeral system with base five, and with place value, might be written.

In a system with base ten we use ten symbols. How many symbols will be needed for a system with base five? It seems that five should be enough. We could invent new ways to write the digits, and give them new names. It will be easier, however, to use the usual digit names: zero, one, two, three, four. The symbols we shall use for base five are: 0, 1, 2, 3, 4. We do not need 5, 6, 7, 8, 9 as you will see.

In base five the numerals for the first four counting numbers are the same as in base ten. Beyond four we make groups of five. The number five will be written 10_{five}. This numeral means "one five and no more." The following examples show how larger numbers can be expressed in base five numerals.

Seven X's are shown at the right. Grouping by fives, we draw a ring around five of the X's. Then we have one group of five and two more.

X X X X X X X
(Seven)

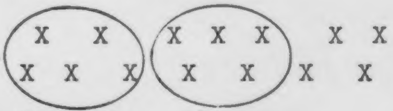


(1 group of five) + (2 more)

This collection of "one five and two more" is written 12_{five}. The "five" must be written to show what base we are using.

Numeral: 12_{five}

The X's in this drawing have already been grouped by fives. How many groups of five are there? How many X's outside of any group?



Numeral: 24_{five}

The table below shows the first ten counting numbers written in numerals of base five.

Numeral in base ten	Number of objects	Numeral in base five	Meaning of base five numerals
1	x	1_{five}	one
2	xx	2_{five}	2 × one
3	xxx	3_{five}	3 × one
4	xxxx	4_{five}	4 × one
5	xxxxx	10_{five}	(1 × five) + (0 × one)
6	xxxxx x	11_{five}	(1 × five) + (1 × one)
7	xxxxx xx	12_{five}	(1 × five) + (2 × one)
8	xxxxx xxx	13_{five}	(1 × five) + (3 × one)
9	xxxxx xxxx	14_{five}	(1 × five) + (4 × one)
10	xxxxx xxxxx	20_{five}	(2 × five) + (0 × one)

In the table we wrote symbols for numbers through ten and had no need for the digits 5, 6, 7, 8, 9.

The numerals 1, 2, 3, 4 have the same meaning in base five as in base ten. This is not true of numerals such as 12. To avoid confusion, a base five numeral has "five" written as in the table.

When no base is written, it is understood that base ten is meant.

For example:

	12	means one ten and two more, or twelve
but	12_{five}	means one five and 2 more, or seven.
	14	means one ten and four more, or fourteen
but	14_{five}	means one five and four more, or nine.

Be careful not to read 14_{five} as "fourteen". Fourteen is the name for (1 × ten) + (4 × 1). The numeral 14_{five} is read simply "one, four, base five." In the same way 20_{five} is read "two, zero, base five"; 33_{five} is read "three, three, base five."

Exercises 2-4a

1. Copy the X's below. Group them by fives, and write the number of X's in base five numerals.

   a. X X X X X X

   b. X X X X X X X X X

   c. X X X X X X
   X X X X X X

   d. X X X X X X
   X X X X X X
   X X X X X X

   e. X X X X X
   X X X X X
   X X X X
   X X X X
   X X X X

2. Draw x's and group them to show the meaning of:

   a. 13_{five}

   b. 21_{five}

   c. 30_{five}

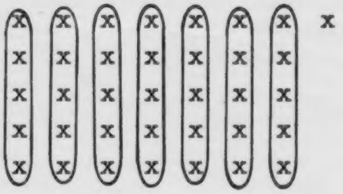
   d. 44_{five}

3. Make a counting table for base five numerals. Copy and complete the table from 1_{five} to 44_{five}. Some parts of the table are completed for you. Use them to check your work.

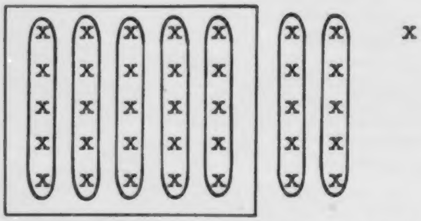
Numeral in base five	Expanded form	Numerals in base ten
1_{five}	(1 × one)	1
2_{five}		
and so on.		
....		
10_{five}	(1 × five) + (0 × one)	5
....		
33_{five}	(3 × five) + (3 × one)	18
....		
44_{five}		

We can use the idea of grouping to help us write numerals larger than 44_{five}.

In the drawing at the right we have shown thirty-six x's. They are grouped in columns of five each. How many fives are there? There are seven. But in base five, we have no digit to show seven. How can we show this number?



In base five we always think of groups of five. We can make a large group of five fives and show this group in a place for five fives (or twenty-five). Our numeral can then be written 121_{five}.



(1 × five × five) + (2 × five) + 1

Exercises 2-4b

Copy the x's shown below. Draw lines to show groups of five x's. When you have five groups, make a larger group with a heavier line. Then write a numeral in base five to show the number of x's.

1.

x x x x x x x x x
x x x x x x x x x
x x x x x x x x
x x x x x x x x
x x x x x x x x

2.

x x x x x x x x
x x x x x x x x
x x x x x x x x
x x x x x x x x
x x x x x x x

3.

x x x x x x x x x x x x
x x x x x x x x x x x
x x x x x x x x x x x
x x x x x x x x x x x
x x x x x x x x x x x

4.

x x x x x x x x x x x x x x x x x
x x x x x x x x x x x x x x x x x
x x x x x x x x x x x x x x x x x
x x x x x x x x x x x x x x x x x
x x x x x x x x x x x x x x x x x

5.

x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x

6. If you always group by fives, how many x's do you need to write to picture the next grouping after:

a. five? b. twenty-five?

7. Suppose that we are making change, and have only pennies, nickels, and quarters. Show how many of each coin are needed for the amounts of change in the table. Use as small a number of coins as you can. Put your answers in a table like the one below.

number of cents	quarters	nickels	pennies
Sample: thirty-two	1	1	2

a. thirteen
b. twenty
c. twenty-nine
d. forty-six
e. fifty-eight
f. seventy-five
g. one hundred sixteen
h. one hundred twenty-four

8. We know that five pennies make one nickel and five nickels make one quarter. Since this is true, we might think of place value in base five in terms of pennies, nickels and quarters. The thirty-two cents of the sample in Question 7 can be written 112_{five}. Write the other parts of Question 7 with base five numerals.

Now we are ready to write place values for the new base five system. How are they different from base ten numerals? In the decimal system the place values are powers of ten. (Starting at the right the places are: 1, 10^1, 10^2, 10^3, and so on.) If five is used as a base, the value of the first place will still be 1, but the second place will be $five^1$, not ten^1. The third place will have a value of $five^2$ (five × five).

What will be the value of the fourth place?

How is this different from the value of the fourth place in base ten?

The table below shows several ways in which the values of places in base five can be expressed.

Place value	five × five × five	five × five	five	one
with exponents	$five^3$	$five^2$	$five^1$	
in words	one hundred twenty-five	twenty-five	five	one
as coins	?	quarters	nickels	pennies

Notice that we do not write (five × five) as (5 × 5). The symbol "5" does not belong in base five numerals.

We wrote decimal numerals in expanded form to show their meaning. We can do the same thing for base five.

$$122_{\text{five}} = (1 \times \text{five} \times \text{five}) + (2 \times \text{five}) + (2 \times \text{one})$$
$$\text{or} \quad (1 \times \text{five}^2) + (2 \times \text{five}^1) + (2 \times \text{one})$$
$$231_{\text{five}} = (2 \times \text{five} \times \text{five}) + (3 \times \text{five}) + (1 \times \text{one})$$
$$\text{or} \quad (2 \times \text{five}^2) + (3 \times \text{five}^1) + (1 \times \text{one})$$

Expanded form helps us find out how a base five numeral is written in our decimal numerals.

$$24_{\text{five}} = (2 \times \text{five}) + (4 \times \text{one})$$
$$\text{In base ten: } = (2 \times 5) + (4 \times 1)$$
$$= 10 + 4 \quad \text{or } 14$$

$$122_{\text{five}} = (1 \times \text{five}^2) + (2 \times \text{five}^1) + (2 \times \text{one})$$
$$\text{In base ten: } = (1 \times 25) + (2 \times 5) + (2 \times 1)$$
$$= 25 + 10 + 2 \quad \text{or } 37$$

Exercises 2-4c

1. Complete: In base five numerals, the place values are all _____ of five.
2. Draw x's and group them to show the meaning of
   a. 14_{five} b. 31_{five} c. 23_{five}
3. Write the decimal numeral for each part of Problem 2.
4. Write each of these numerals in expanded form, using exponents. Then find its value in decimal numerals.
   a. 13_{five} d. 40_{five} g. 222_{five}
   b. 24_{five} e. 123_{five} h. 403_{five}
   c. 32_{five} f. 312_{five} i. 2134_{five}
5. In the numeral 23_{five} the "3" represents three ones or (3 × one). What does the "3" represent in each of the following?
   a. 301_{five} b. 132_{five} c. 223_{five} d. 3420_{five}

6. In decimal numerals the name "forty" means four tens, "fifty" means five tens, and so on. Try to invent some number names that could be used for the following base five numerals.

   a. 20_{five} b. 30_{five} c. 40_{five} d. 100_{five}

7. When we count in base five, the numeral following 4_{five} is 10_{five}. Write the numeral which follows each of these:

   a. 444_{five} b. 404_{five} c. 3444_{five}

8. All the numerals in the following paragraph are written in base five. Rewrite the paragraph changing to decimal numerals.

   Alan was 23 years old on Friday. He had a birthday party to which 40 guests were invited. It was a fine party. The guests ate 222 hamburgers, 143 doughnuts, 12 quarts of ice cream and 200 bottles of pop. At 13 o'clock they all went to a show. Tickets cost 302 cents apiece.

*9. a. Which of the following base ten numerals represent even numbers? 6, 11, 14, 23, 35, 32, 40

   b. What is an easy way of deciding whether a base ten numeral represents an even number?

   c. Which of the following base five numerals represent even numbers?

   4_{five} 11_{five} 14_{five} 23_{five} 22_{five}

   32_{five} 111_{five} 123_{five} 100_{five}

   d. What is an easy way of deciding whether a base five numeral represents an even number?

   e. Write several more base five numerals and test your answer for d.

*10. a. Which of the following base ten numerals represent numbers which are divisible by ten? (Divisible by ten means that when we divide by ten there is no remainder.)

   5, 11, 20, 26, 30, 43, 50, 54

   b. What is an easy way of deciding whether a base ten numeral represents a number that is exactly divisible by ten?

c. Which of the following base five numerals represent numbers which are divisible by five?

4_{five} 11_{five} 20_{five} 23_{five} 30_{five} 40_{five}

d. What is an easy way of deciding whether a base five numeral represents a number divisible by five?

e. Explain why your answer to d. is true.

2-5. Addition and Subtraction in Base Five.

Now that you can count and write numerals in base five, try some computation.

Notice how much the addition problem in base ten below looks like the second problem written in base five.

Base Ten	Base Five
$\begin{array}{r} 13 \\ 21 \\ \hline 34 \end{array}$	$\begin{array}{r} 13_{five} \\ 21_{five} \\ \hline 34_{five} \end{array}$

Does $3_{five} + 1_{five} = 4_{five}$?

Does $2_{five} + 1_{five} = 3_{five}$?

Does 34_{five} have the same meaning as 34?

If you write 34_{five} in expanded form, you find it means $(3 \times \text{five}) + (4 \times 1)$ or 19, not 34.

Were the numbers added (addends) the same in the two problems?

The two problems above were easy to do because there was no need to "carry." It will be helpful to have some tables of basic facts in addition before trying harder problems.

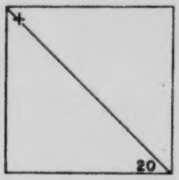
Exercises 2-5a

1. Using cross-ruled paper, copy and complete the following addition table for base ten numerals:

Addition, Base Ten

+	0	1	2	3	4	5	6	7	8	9	10
0	0										
1	1	2									
2			4								
3	3	4	5	6							
4	4	5	6	7	8						
5	5	6	7	8	9	10	11				
6											
7											
8											
9									17		
10											

2. a. Draw a line from the upper left corner to the lower right corner of the table as shown at the right. What do you notice about the numerals in the two parts?



b. When you use the addition table in an exercise, such as 7 + 9, find the first number, 7, in the column at the left and the second number in the top row. Then the arrows in the table in Problem 1 show 9 + 8, not 8 + 9. Use the table in this way to find the sum of: 7 + 9; 9 + 7; 7 + 6; 6 + 7.

c. What does the table show about the order of adding two numbers?

3. Using cross-ruled paper, copy and complete the following addition table in base five numerals:

Addition, Base Five

+	0	1	2	3	4	10
0						
1			3			
2				10		
3					12	
4						
10						

Do not try to memorize the table.

4. a. Draw a line from the upper left corner to the lower right corner as you did in Problem 1.

b. What do you notice about the numerals in the two parts of this table?

c. Using the table in the way suggested in 2b above, show that $3_{five} + 4_{five} = 4_{five} + 3_{five}$.

5. Your table for base ten shows that $10 + 3 = 13$. The table for base five shows that $10_{five} + 3_{five} = 13_{five}$.

   In both cases the symbols are the same. Why do you think this is true?

6. Why is the numeral in the lower right corner of the table for base ten exactly like the one in the same corner for base five? Does 20_{five} mean the same as 20?

Look at the problem below this paragraph. You would do it as shown at the left side of the page, and you would think of "carrying" one ten to the tens column. The problem is rewritten at the right of the page to show how this regrouping (carrying) method works.

36		36 = 3 tens + 6 ones
$\underline{27}$		$\underline{27 = 2 \text{ tens} + 7 \text{ ones}}$
63		5 tens + 13 ones
	Regrouping:	6 tens + 3 ones = 63

Here is another problem in base five. Try to find the answer to the problem written at the left before you read the explanation.

23_{five}		23_{five} = 2 fives + 3 ones
$\underline{14_{five}}$		$\underline{14_{five} = 1 \text{ five} + 4 \text{ ones}}$
		3 fives + seven ones
	Regrouping:	4 fives + 2 ones = 42_{five}

The part of the problem most important for you to understand is the very last step. What really happens in the regrouping process is this:

3 fives + seven ones
= 3 fives + 1 $\underline{\text{five}}$ + 2 ones
= 4 fives + 2 ones.

Addition problems can be checked by changing numerals in base five to base ten as below.

$$\begin{array}{rcr}
 & & \underline{\text{Check}} \\
23_{\text{five}} & = & 13 \\
+\ \underline{14_{\text{five}}} & = & \underline{9} \\
42_{\text{five}} & & 22
\end{array}$$

Does 13 + 9 = 22?

Does $42_{\text{five}} = 22$?

Do the sums agree?

Exercises 2-5b

1. Add the following. Use your addition table in base five numerals to help you. Check by changing to base ten.

Example:

$$\begin{array}{rcll}
 & & \underline{\text{Check}} & \\
222_{\text{five}} & = & (2 \times 25) + (2 \times 5) + (2 \times 1) & = \quad 62 \\
+\ \underline{124_{\text{five}}} & = & (1 \times 25) + (2 \times 5) + (4 \times 1) & = \quad \underline{39} \\
401_{\text{five}} & = & (4 \times 25) + (0 \times 5) + (1 \times 1) = 101 & \longleftrightarrow 101
\end{array}$$

1. $\begin{array}{r} 21_{\text{five}} \\ \underline{13_{\text{five}}} \end{array}$

2. $\begin{array}{r} 33_{\text{five}} \\ \underline{4_{\text{five}}} \end{array}$

3. $\begin{array}{r} 24_{\text{five}} \\ \underline{14_{\text{five}}} \end{array}$

4. $\begin{array}{r} 202_{\text{five}} \\ \underline{132_{\text{five}}} \end{array}$

5. $\begin{array}{r} 122_{\text{five}} \\ \underline{232_{\text{five}}} \end{array}$

6. $\begin{array}{r} 224_{\text{five}} \\ \underline{121_{\text{five}}} \end{array}$

7. $\begin{array}{r} 14_{\text{five}} \\ 23_{\text{five}} \\ 11_{\text{five}} \\ \underline{22_{\text{five}}} \end{array}$

8. $\begin{array}{r} 21_{\text{five}} \\ 30_{\text{five}} \\ 44_{\text{five}} \\ \underline{31_{\text{five}}} \end{array}$

9. Subtract by grouping as in the example.

$$\begin{array}{rcl} 13_{\text{five}} & = & \text{xxxxx xxx} \\ -\ 4_{\text{five}} & = & \underline{\qquad\qquad\text{xxxx}} \\ \hline & & \qquad\qquad\text{xxxx} \text{ or } 4_{\text{five}} \end{array}$$

a. $\begin{array}{r} 11_{\text{five}} \\ -\ 2_{\text{five}} \\ \hline \end{array}$

c. $\begin{array}{r} 10_{\text{five}} \\ -\ 3_{\text{five}} \\ \hline \end{array}$

b. $\begin{array}{r} 12_{\text{five}} \\ -\ 4_{\text{five}} \\ \hline \end{array}$

d. $\begin{array}{r} 21_{\text{five}} \\ -\ 13_{\text{five}} \\ \hline \end{array}$

10. Use your addition table to check the answers for a, b, c, above.
Then use the table to get the answers for these.
Counting is "no fair." Use the table!

a. $\begin{array}{r} 12_{\text{five}} \\ -\ 3_{\text{five}} \\ \hline \end{array}$

b. $\begin{array}{r} 10_{\text{five}} \\ -\ 1_{\text{five}} \\ \hline \end{array}$

c. $\begin{array}{r} 11_{\text{five}} \\ -\ 4_{\text{five}} \\ \hline \end{array}$

Take a close look at regrouping, or "borrowing" in subtraction. Subtraction is usually done without writing all the steps shown in the example, but the steps will help you see what must be done when base five numerals are used. Subtract:

$$\begin{array}{rlcl} 32 & \text{means 3 tens + 2 ones} & = & \text{2 tens + 12 ones} \\ \underline{14} & \text{means 1 ten + 4 ones} & = & \underline{\text{1 ten + 4 ones}} \\ & & & \text{1 ten + 8 ones or 18} \end{array}$$

Where did the "12 ones" in the top line of the example come from?

A similar plan is used when subtraction with base five numerals is done:

$$\begin{array}{rlcl} 32_{\text{five}} & \text{means 3 fives + 2 ones} & = & \text{2 fives + } 12_{\text{five}} \text{ ones} \\ \underline{14_{\text{five}}} & \text{means 1 five + 4 ones} & = & \underline{\text{1 five + 4 ones}} \\ & & & \text{1 five + 3 ones or } 13_{\text{five}} \end{array}$$

Exercises 2-5c

1. Subtract, and check as you did in addition. Note the following example:

$$\begin{array}{lcll} & & & \text{Check in Base Ten} \\ 41_{five} & = & (4 \times 5) + (1 \times 1) & = 21 \\ \underline{13_{five}} & = & (1 \times 5) + (3 \times 1) & = \underline{8} \\ 23_{five} & & (2 \times 5) + (3 \times 1) & = 13 \longleftrightarrow 13 \end{array}$$

You may not need to write all the steps that are in the example.

a. $\begin{array}{r} 32_{five} \\ \underline{24_{five}} \end{array}$

d. $\begin{array}{r} 302_{five} \\ \underline{142_{five}} \end{array}$

b. $\begin{array}{r} 43_{five} \\ \underline{14_{five}} \end{array}$

e. $\begin{array}{r} 100_{five} \\ \underline{23_{five}} \end{array}$

c. $\begin{array}{r} 231_{five} \\ \underline{104_{five}} \end{array}$

f. $\begin{array}{r} 310_{five} \\ \underline{134_{five}} \end{array}$

2. How could you check this subtraction problem with decimal numerals?

$$\begin{array}{r} 63 \\ \underline{-\ 27} \\ 36 \end{array}$$

3. Try to check the a, b, c, parts of Problem 1 in the same way. Play fair! Be sure they do check.

2-6. Multiplication in Base Five.

Exercises 2-6a

1. a. In multiplication a table of basic facts is useful. Using cross-ruled paper, copy and complete the multiplication table for base ten. You should know all the numbers for this table from memory.

Multiplication, Base Ten

×	0	1	2	3	4	5	6	7	8	9	10
0											
1											
2											
3											
4											
5											
6											
7											
8											
9								63			
10											

b. Draw a diagonal line from the upper left to the lower right corners. What do you notice about the two parts of the table?

c. Use the table in the way suggested for addition: in an exercise, 9 × 7, the first numeral is in the left column; the second numeral is in the top row. So the dotted lines of the table show the product 9 × 7, rather than 7 × 9.

Does 9 × 7 = 7 × 9?
Does 5 × 8 = 8 × 5?

d. What number multiplied by 6 gives a product of 24? Can the table be used to find answers in division?

2. Show with x's each of the numbers below. Group them by fives, and write the base five numeral.

Example: Two fours = xxxx xxxx
= $\boxed{\text{xxxx x}}$xxx or 13_{five}

a. two threes b. three fours c. four fours

3. a. On cross-ruled paper copy and complete the multiplication table for base five. Do not try to memorize it. The value of the table lies in your understanding of it.

Multiplication, Base Five

×	0	1	2	3	4	10
0						
1						
2				11		
3					22	
4			13			
10						

b. Draw a diagonal line as in 1b above. What can you say about the two parts of the table?

c. Does $3_{five} \times 4_{five} = 4_{five} \times 3_{five}$?

d. From the table, what is $13_{five} \times 4_{five}$? $32_{five} \times 3_{five}$?

4. If you had not memorized any of the numbers in either of the multiplication tables, which would be easier to learn, the table for base ten, or for base five? Why?

Compare the two multiplication problems below:

Base Ten

$$\begin{array}{r} 22 \\ \times\ 2 \\ \hline 44 \end{array}$$

Base Five

$$\begin{array}{r} 22_{five} \\ \times\ 2_{five} \\ \hline 44_{five} \end{array}$$

Does 44 mean the same as 44_{five}?

Sometimes it is necessary to regroup in multiplication. The examples below show how this is done, first in base ten, then in base five numerals.

Base Ten

$$\begin{array}{r} 44 \\ \times\ 3 \\ \hline 132 \end{array}$$

Or showing each product without regrouping (carrying)

Base Ten

$$\begin{array}{rr} & 44 \\ & \times\ 3 \\ \hline 3 \times 4 = & 12 \\ 3 \times 4 \text{ tens} = & 12 \\ \hline & 132 \end{array}$$

Base Five

$$\begin{array}{r} 44_{five} \\ \times\ 3_{five} \\ \hline 242_{five} \end{array}$$

Or, showing each product

Base Five

$$\begin{array}{rr} & 44_{five} \\ & \times\ 3_{five} \\ \hline 3 \times 4 = & 22_{five} \\ 3 \times 4 \text{ fives} = & 22_{five} \\ \hline & 242_{five} \end{array}$$

After a little practice you will find that "carrying" in base five numerals can be done quickly.

*Exercises 2-6b

Multiply, using the following base five numerals.

Check

$$\begin{array}{rlcr}
44_{\text{five}} & = (4 \times 5) + (4 \times 1) & = & 24 \\
\times\ \ 3_{\text{five}} & & & \times\ 3 \\
\hline
242_{\text{five}} & = (2 \times 25) + (4 \times 5) + (2 \times 1) & & 72 \\
 & = \quad 50 \quad + \quad 20 \quad + \quad 2 & = & 72
\end{array}$$

1. $\begin{array}{r} 13_{\text{five}} \\ \times\ 3_{\text{five}} \\ \hline \end{array}$

2. $\begin{array}{r} 24_{\text{five}} \\ \times\ 3_{\text{five}} \\ \hline \end{array}$

3. $\begin{array}{r} 43_{\text{five}} \\ \times\ 2_{\text{five}} \\ \hline \end{array}$

4. $\begin{array}{r} 34_{\text{five}} \\ \times\ 3_{\text{five}} \\ \hline \end{array}$

5. $\begin{array}{r} 124_{\text{five}} \\ \times\ 3_{\text{five}} \\ \hline \end{array}$

6. $\begin{array}{r} 203_{\text{five}} \\ \times\ 4_{\text{five}} \\ \hline \end{array}$

7. $\begin{array}{r} 322_{\text{five}} \\ \times\ 3_{\text{five}} \\ \hline \end{array}$

*8. $\begin{array}{r} 43_{\text{five}} \\ \times\ 23_{\text{five}} \\ \hline \end{array}$

*9. $\begin{array}{r} 324_{\text{five}} \\ \times\ 42_{\text{five}} \\ \hline \end{array}$

*2-7. Division in Base Five.

Division is left as an exercise for you. The procedure is the same as for decimal numerals.

First, do Problem 1 of Exercises 2-7a. Be sure you know how to use the multiplication table to help you in division. Then study the two division problems below this paragraph. All the numerals are written in base five.

$$\begin{array}{r} 42 \\ 3\,\overline{)231} \\ \underline{22} \\ 11 \\ \underline{11} \end{array} \qquad\qquad \begin{array}{r} 213 \\ 12\,\overline{)3111} \\ \underline{24} \\ 21 \\ \underline{12} \\ 41 \\ \underline{41} \end{array}$$

In the first problem:

Do you see how "4" in the quotient is found?
Where did the "22" come from?
How was the "2" in the quotient found?

In the same way, study the second problem. Then do the rest of the exercises.

*Exercises 2-7a

1. Use the multiplication table for base five to do the following division exercises.

a. $11_{five} \div 3_{five}$ d. $24_{five} \div 3_{five}$
b. $13_{five} \div 2_{five}$ e. $13_{five} \div 3_{five}$
c. $31_{five} \div 4_{five}$ f. $34_{five} \div 4_{five}$

2. Complete the divisions. All the numerals are written in base five.

a. $2\,\overline{)132}$ d. $13\,\overline{)404}$
b. $4\,\overline{)211}$ e. $24\,\overline{)2014}$
c. $3\,\overline{)1333}$

Review Exercises 2-7b

1. Write in expanded form:

   a. 302_{five} b. 167_{ten}

2. Which of the numerals in Problem 1 represents the larger number?

3. Add the following:

   a. $42_{\text{five}} + 14_{\text{five}}$ b. $\begin{array}{r} 203_{\text{five}} \\ \underline{132_{\text{five}}} \end{array}$

4. Subtract:

   a. $30_{\text{five}} - 2_{\text{five}}$ b. $222_{\text{five}} - 4_{\text{five}}$

5. Multiply:

   a. $23_{\text{five}} \times 4_{\text{five}}$ b. $43_{\text{five}} \times 3_{\text{five}}$

6. Rewrite the following paragraph replacing the base five numerals with decimal numerals.

   Louise takes seventh grade mathematics in room 443_{five}. The book she uses is called Junior High School Mathematics 12_{five}. It has 30_{five} chapters and 3034_{five} pages. There are 112_{five} pupils in the class which meets 10_{five} times each week for 210_{five} minutes daily. 23_{five} of the pupils are girls and 34_{five} are boys. The youngest pupil in the class is 21_{five} years old and the tallest is 231_{five} inches tall.

7. 122_{five} members of Boy Scout Troop 1021_{five} went to a baseball game last week. During the game they ate 322_{five} hotdogs, 214_{five} sodas, and 103_{five} candy bars. The total paid attendance for the game was 101103_{five}. Everyone had a good time since the home team won by a score of 23_{five} to 14_{five}.

   a. Express each of the numerals above in base ten.
   b. Express the sum of the hotdogs and sodas in base five.
   c. Express your answer to part (b) in base ten.
   d. Express the total number of runs scored in the game in base five. In base ten.

2-8. Changing Decimal Numerals to Base Five Numerals.

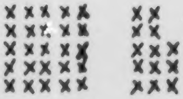
You have learned how to change a number written in base five numerals to decimal numerals. Let us see how to change from base ten to base five.

In base five the values of the first four places are shown in the table:

Place value	five × five × five	five × five	five	one
With exponents:	$(\text{five})^3$	$(\text{five})^2$	five^1	one
In base ten:	125	25	5	1
In base five:	1000_{five}	100_{five}	10_{five}	1_{five}

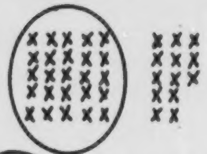
Suppose we wish to change 38_{ten} to base five numerals. Below are listed the steps to be followed.

A. Consult the table of place values to find the size of the largest group (power of five) contained in 38. Since 38 is more than 25 but less than 125, the "five^2" place will be the largest place needed.



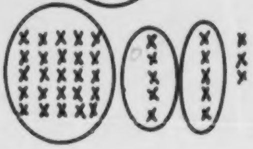
B. Find how many of these groups (25's) are contained in 38. Dividing, we find

$$\begin{array}{r} 1 \text{ R } 13 \\ 25\,\overline{)\,38} \\ \underline{25} \\ 13 \end{array}$$



C. Look at the remainder from the division. Does it contain any of the next smaller group, fives? Divide to find out how many.

$$13 \div 5 = 2 \text{ R } 3$$



D. Continue in the same way. Look at the remainder to find if it contains any of the next smaller group. In this case it contains 3 ones. Now you can write the numeral in base five:

38_{ten} = 1 twenty-five + 2 fives + 3 ones, or 123_{five}.

Try another problem: change 352_{ten} to base five.

A. Consult the table of place values. 352 is larger than 125. Then it will require at least the fourth place (five^3).

B. Find how many of these groups (125's) are contained in 352.

$$\begin{array}{r} 2\ \text{R}\ 102 \\ 125\ \overline{)\,352} \\ \underline{250} \\ 102 \end{array}$$

There are 2 groups of 125.

C. Look at the remainder, 102. What is the size of the next smaller group? It is five^2 (or 25). How many 25's are there in 102?

$$\begin{array}{r} 4\ \text{R}\ 2 \\ 25\ \overline{)\,102} \\ \underline{100} \\ 2 \end{array}$$

There are 4 groups of 25.

D. Look at the remainder, 2. What is the size of the next smaller group? It is "five" but 2 does not contain any fives. Our numeral therefore must show zero fives, and 2 ones.

Then $352 = (2 \times \text{five}^3) + (4 \times \text{five}^2) + (0 \times \text{five}) + (2 \times 1)$

$= 2402_{\text{five}}$.

Exercises 2-8

1. Show that:

   a. $17 = 32_{\text{five}}$

   b. $36 = 121_{\text{five}}$

   c. $68 = 233_{\text{five}}$

   d. $75 = 300_{\text{five}}$

   e. $92 = 332_{\text{five}}$

   *f. $183 = 1213_{\text{five}}$

2. Change the following decimal numerals to base five numerals.

   a. 14

   b. 23

   c. 37

   d. 56

   e. 81

   *f. 127

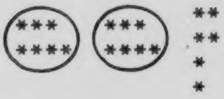
   *g. 262

2-9. Numerals In Other Bases.

After your work with base five numerals, you know that it is possible to express numbers in systems other than the decimal scale. The exercises below will help you with numerals in a variety of different systems.

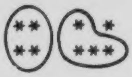
Exercises 2-9a

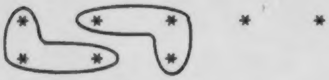
1. The *'s at the right are grouped in sevens.

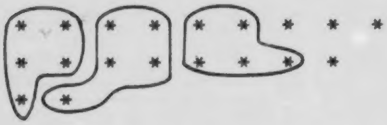


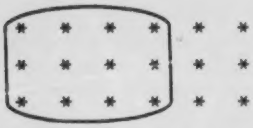
   a. Explain why the numeral for the number of *'s can be written 26_{seven}.
   b. Tell how to read 26_{seven}.
   c. Write the decimal name for the same number.

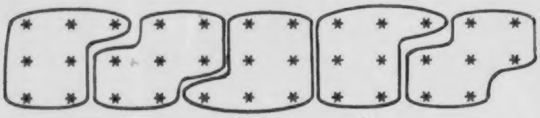
Notice the ways the stars are grouped and write the numeral. Be sure to indicate the base in each case.

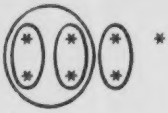
Example:  21_{four}

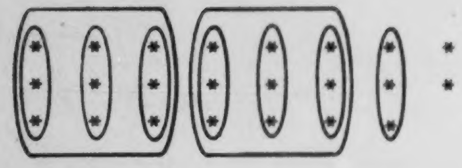
2.  ________

3.  ________

4.  ________

5.  ________

6.  ________

7.  ______

In the following exercises group the stars as indicated above and write the numeral in base 5, base 7, and base 4.

Example:	Base 5	Base 7	Base 4
* * * * * * * * * * * * *	23_{five}	16_{seven}	31_{four}

8. * * * *
*

9. * * * * * * * *

10. * * * * * *
* * * * *

11. * * * * * * *
* * * * * * * *

12. * * * * * * * *
* * * * * * * *
* * * * * * * *

Base Seven

The exercises below deal with base seven numerals. If you understood your work with base five numerals, these will be fun for you.

Exercises 2-9b

1. Group the stars at the right in sevens and write the base seven numeral.

****
****
******

2. Draw *'s to show the numbers represented by

a. 11_{seven} b. 25_{seven}

3. Express in base seven notation the numbers from one through twenty-five. On your paper complete the table started below:

Base ten	1	2	3	4	5	6	7	8	9	10	...	18	...	22	...	25
Base seven	1	2								13		24		31		

4. What is the decimal numeral for 100_{seven}?
5. Which of the following represent even numbers?

   a. 15_{seven} b. 111_{seven} c. 30_{seven}
6. How many sevens are in 60_{seven}?
7. Try some computation in base seven numerals.

   a. Add: $$\begin{array}{r} 42_{seven} \\ \underline{13}_{seven} \end{array}$$
   b. Add: $$\begin{array}{r} 34_{seven} \\ \underline{62}_{seven} \end{array}$$
   c. Subtract: $$\begin{array}{r} 62_{seven} \\ \underline{15}_{seven} \end{array}$$
8. Write the decimal names for the place values in the seven system for four places, beginning with the one place.

Base Twelve

Some people think that a twelve system is better than a decimal system. They say that twelve can be divided exactly by 2, 3, 4, and 6 as well as by 1 and 12. To them ten is a poor choice because it can be divided exactly by fewer numbers. Also, twelve is widely used in business where many things are bought by the dozen or the gross. A gross is a dozen dozen. A gross of objects can be packed in a carton in many more ways than one hundred such objects. Twelve appears in measurement too, for twelve inches make a foot. The twelve system is called the duodecimal system.

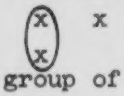
In the twelve system twelve symbols are needed. It is necessary to invent two new symbols. T is often used for ten and E for eleven. Then counting is carried out as follows.

Base ten	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Base twelve	1	2	3	4	5	6	7	8	9	T	E	10	11	12

Exercises 2-9c

1. Continue to count in base twelve to fifty.
2. What is the decimal name for 100_{twelve}?
3. What is the base twelve numeral for 100?
4. What is the decimal name for
   a. 6E? b. T5? c. TE? d. 4T?
5. Which system expresses the largest number with the fewest symbols: base five, base seven, base ten, or base twelve?
6. Name the place values in the duodecimal system for three places beginning with one.

Base Two

History tells about early people who used a system based on two for writing numerals. Some Australian tribes still count by pairs, "one, pair, pair and one, pair of pairs," and so on. The two system of numerals is called the binary system. It groups by pairs as in the diagram.  How many groups of two are shown? Three x's mean 1 group of two and 1 one. In binary notation 11_{two} means three.

The binary system uses only two symbols 0 and 1. Some of the place values in the binary system are:

$$|\,\text{two}^5\,|\,\text{two}^4\,|\,\text{two}^3\,|\,\text{two}^2\,|\,\text{two}\,|\,\text{one}\,|$$

Counting starts as follows:

Decimal	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Binary	1	10	11	100	101	110	111	1000	?	?	?	?	?	?

Modern high speed computers are electrically operated. A simple electric switch has only two positions, open (on) or closed (off). Computers operate on this principle. Because there are only two positions for each place, the computers use the binary system of notation.

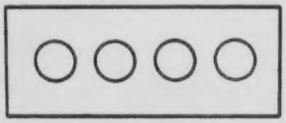


Figure A

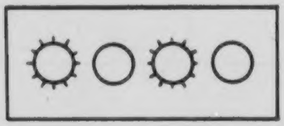


Figure B

We will use the drawing at the right to represent a computer. The four circles represent four lights on a panel, and each light represents one place in the binary system. When the current is flowing, the light is on, shown in Figure B as ☼. A ☼ is represented by the symbol "1." When the current does not flow, the light is off, shown by ○ , in Figure B. This is represented by the symbol "0." The panel in Figure B represents the binary numeral 1010_{two}. What decimal numeral is represented by this numeral? The partially completed table shows the place values for the powers of two.

Place Value	two^4	two^3	two^2	two^1	one
In decimal	2 × 2 × 2 × 2	2 × 2 × 2	2 × 2	2	1
Base Ten	16	8	4	2	1
Base Two	$10{,}000_{\text{two}}$	1000_{two}	100_{two}	10_{two}	1_{two}

$$\begin{aligned} 1010_{\text{two}} &= (1 \times \text{two}^3) + (0 \times \text{two}^2) + (1 \times \text{two}^1) + (0 \times \text{one}) \\ &= (1 \times 8) + (0 \times 4) + (1 \times 2) + (0 \times 1) \\ &= 10_{\text{ten}} \end{aligned}$$

Exercises 2-9d

1. How many symbols are needed in each system?

   a. Duodecimal
   b. Base five
   c. Base Seven
   d. Decimal
   e. Binary

2. What is the smallest possible number of symbols necessary for a numeration system using place value?

3. What is the meaning of $10 \times 10 = 100$ in the
   a. Binary system? b. Base five system?
   c. Duodecimal system?
4. Make an addition chart for the binary system:
   a.

+	0	1
0		
1		

   b. How many addition facts are there?
5. Make a multiplication chart for base 2.
   a.

X	0	1
0		
1		

   b. How many multiplication facts are there?
   c. How do the addition and multiplication charts compare?
   d. Which seems to be easier, memorizing the addition and multiplication facts in the binary system or in the decimal system?
   e. Why do you suppose people would be slow to adopt the binary system in place of the decimal system?

6. Add the following:

a. $\begin{array}{r} 101_{two} \\ \underline{10_{two}} \end{array}$ b. $\begin{array}{r} 111_{two} \\ \underline{101_{two}} \end{array}$

7. Make a table of numerals as shown below:

Base ten	Base two	Base five	Base eight
1			
2			
5			
7			
15			
16			
32			
64			
256			

*8. Write the following numerals in the indicated ways:

a. 11000_{two} ____________ten ____________seven

b. 25_{six} ____________five ____________twelve

c. 68_{nine} ____________ten ____________four

d. 27_{ten} ____________two ____________three

9. BRAINBUSTER. Study the following table of numerals in base two and base eight.

$$\begin{array}{lcccccccc} \text{Base Two} & 1 & 10 & 11 & 100 & 101 & 110 & 111 & \underset{\downarrow}{1}\underbrace{000} \\ \text{Base Eight} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1\ \ 0 \end{array}$$

$$\begin{array}{lcccccc} \text{Base Two} & \underset{\downarrow}{1},\underbrace{001} & \underset{\downarrow}{1},\underbrace{010} & \underset{\downarrow}{1},\underbrace{011} & \underset{\downarrow}{1},\underbrace{100} & \underset{\downarrow}{1},\underbrace{101} & \underset{\downarrow}{1},\underbrace{110} \\ \text{Base Eight} & 1\ \ 1 & 1\ \ 2 & 1\ \ 3 & 1\ \ 4 & 1\ \ 5 & 1\ \ 6 \end{array}$$

You will see that a whole group of three digits in the binary system can be expressed by one digit in the base eight system. For example:

$$\begin{array}{rccc} & 101, & 011, & 010_{two} \\ = & 5 & 3 & 2_{eight} \end{array}$$

People who work with computers find this helpful. It is easier to remember 532_{eight} than $101,011,010_{\text{two}}$.

Write the value of $111,010,110_{\text{two}}$ in base eight numerals.

10. BRAINBUSTER. a. An inspector of weights and measures carries a set of weights which he uses to check the accuracy of scales. What is the smallest number of weights the inspector may have in his set, and what are these weights, if he is to check the accuracy of scales from 1 ounce to 15 ounces?

b. From 1 ounce to 31 ounces?

2-10. Summary.

The decimal system is the result of efforts of men over thousands of years to develop a useful system of writing numerals. It is not a perfect system, but in some ways it is better than other systems. In this chapter you have studied some of the ancient systems which, in their time, represented great progress. You have also studied other systems in different bases. It is not expected that you will use a base other than ten in your everyday living. You studied other systems to gain a better understanding of your own numerals.

You have learned that a number may be expressed in different numerals. For example, twelve may be written as ∩||, XII, 12_{ten}, or 22_{five}, and so on. These numerals are not the same, yet they represent the same number. Be sure you understand that the symbols we may use are not the numbers themselves: "XII" is not a number, nor is "22_{five}." They are only different numerals, or names, or marks made on paper, that stand for a particular number. In the same way the word "pencil" is not the same as the object you hold in your hand when you are writing on paper.

The Egyptians might not have known that their system of numerals was based on ten. To know this, they would have had to know that it is possible to use other bases for a numeration system. You know this now, and you know that it is possible to use any whole number greater than one as a base. Some of these numeration systems have a practical use, as the binary system is used in electric computers. The machine operates electrically with a large "memory" of stored facts. Man was able to invent modern high speed computers because he had invented a system of representing numbers that could be operated by electricity.

Some of the important terms introduced in this chapter follow.

BASE: 1. In the expression 10^4, the base is ten. The whole expression 10^4 is the fourth power of ten.

2. The number which gives the size of the smallest group in a system of writing numbers. If five is the base, the smallest group is five and the first four place values are:

$$|\text{five}^3 \mid \text{five}^2 \mid \text{five} \mid \text{one}|$$

3. The numeral for the base itself in any system is 10.

BINARY SYSTEM: The system of writing numerals in base two.

DECIMAL SYSTEM: The system of writing numerals in base ten.

DIGIT: Any one of the single symbols which are used in a system of writing numerals. Digits of the decimal system are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

DUODECIMAL SYSTEM: The system of writing numerals in base twelve.

EXPANDED FORM: An expression for a number which shows how it is built on powers of the base. In expanded form:

$$324_{\text{ten}} = (3 \times \text{ten}^2) + (2 \times \text{ten}) + (4 \times 1)$$

$$324_{\text{five}} = (3 \times \text{five}^2) + (2 \times \text{five}) + (4 \times 1)$$

EXPONENT: A number symbol written at the right and above another number symbol. It shows how many times the base is to be used as a factor. (See Power.)

FACTOR: In multiplication, any one of the numbers that form a product. In (8×5), 8 is a factor and 5 is a factor. In $(7 \times 3 \times 12)$, 7 is a factor, 3 is a factor, and 12 is a factor.

NUMERAL: A symbol used to represent a number. IV, 4, 1960 are numerals.

PLACE VALUE: The value assigned to a position in a system of writing numerals. Some of the place values for the decimal system and the base five system are:

Base Ten	one thousand	one hundred	ten	one
Base Five	one hundred twenty-five	twenty-five	five	one

POWER: 10^4 (or $10 \times 10 \times 10 \times 10$), 10^3 (or $10 \times 10 \times 10$) are powers of ten. 5^2, 5^4, 5^9 are all powers of five. In such expressions, the number used as a factor is the base; the small numeral showing the number of factors is the exponent. In 10^4, the base is 10, the exponent is 4.

SYMBOL: A mark of any sort which stands for a number, operation, relation, and so on. Some symbols used in mathematics are 3, VII, ÷ , = .

2-11. Chapter Review.

Exercises 2-11

1. Write the following numerals in words.
   a. 2,035 b. 56,208 c. 876,500,210
2. Change the Roman numerals to decimal numerals.
   a. XXXII b. CCLI c. XIX d. CM
3. How many number symbols are needed to write numerals in a base eight system?

4. Write in expanded form, using exponents.
   a. 2314
   b. 1304_{five}
   c. 111_{two}
   d. 126_{seven}
5. What is the base of the decimal system?
6. Which of these statements are true?
   a. A football team has 21_{five} players.
   b. There are 22_{five} months in a year.
   c. October has 111_{five} days.
   d. $4 \times 4 = 16_{five}$
   e. A dollar is worth 400_{five} cents.
7. Find the value of each of these expressions.
   a. 5^4 b. 2^4 c. 7^2 d. 3^3
   e. 4 to the third power
   f. 10 to the first power
   g. 2 to the fifth power.
8. In Lincoln's Gettysburg Address, the term "four score and seven" is used. What number base does this represent? Write the decimal numeral for this number.
9. What base was used in writing the numerals below?
   a. $x\ x\ x\ x\ x\ x = 20_?$
   b. $x\ x\ x\ x\ x\ x\ x\ x = 12_?$
   c. $x\ x\ x\ x\ x\ x\ x\ x\ x\ x\ x\ x = 22_?$
   d. $4 + 3 = 11_?$
   e. $5 \times 4 = 22_?$
10. What number base has been used to write "fifteen" in each of the numerals below?
   a. 13 b. 21 c. 30 d. 1111
11. BRAINBUSTER. A numeration system with place value uses the symbols O, A, B, C, and D to represent numbers from zero to four. Write the decimal numeral for DCBAO.
12. BRAINBUSTER. Using the symbols from Problem 11, count from one to twenty.

Chapter 3

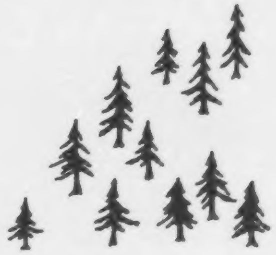
WHOLE NUMBERS

3-1. Introduction.

There is more to mathematics than adding, subtracting, multiplying, and dividing. "Is that so!", you say, "That's all I've ever done." You are probably right! This year we are going to try a new path in mathematics. Let's call it *the path to understanding mathematics*. Not everything will be new and there will be practice with basic skills. The football players on your team practice running, kicking, and throwing every day. Daily practice on the basic skills of mathematics is necessary, too.

Many of the ideas we will work with are already familiar to you. Some of the ideas will seem strange at first. Many mathematicians, however, believe that these ideas will help you to understand mathematics better. *Understanding* is the key to easier learning.

3-2. Sets.

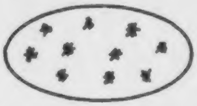


If you wished to describe the forest in the figure, how would you do it? Would you call it a "bunch" of trees? A "group" of trees? A "collection" of trees? Any of these words might be used to describe the forest.

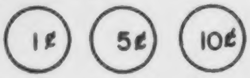
In mathematics we often need to talk about collections of things. Some of the things we say about collections refer to the collection as a whole and not to the separate things which make up the collection. For example, when we say "the forest is large" we are talking about the forest and not the individual trees. When we say that the trees are large, we are talking about the individual trees and not about the forest.

Mathematicians have agreed to use the word *set* whenever they want to talk about collections or groups of things, such as:

A set of numbers: 5, 36 7, 8
A set of boys: John, Tom, Ed
A set of stars in a diagram:



A set of coins:



In the first example, we call 5, 36, 7, and 8 members of the set. In mathematics, the members of the set are called ELEMENTS. Thus, each star in the third example is an ELEMENT of the set of stars in the diagram. Each coin in the last example is an element of a set of coins.

Exercises 3-2a
(Class Discussion)

1. Name some other words which stand for collections. Show how each word is used. Example: The word "deck" as in a "deck of cards."
2. Give other examples of sets which you can see in your classroom.
3. Give examples of sets which are in your home.

We can represent a set by placing braces, { }, around its elements. In this way we can write:

A set of presidents: {Eisenhower, Washington, Lincoln, Grant}
A set of books: {English, mathematics, history, spelling, science}
A set of makes of cars: {Ford, Chevrolet, Plymouth}
A set of numbers: {1, 2, 3, 4, 5, 6, 7, 8}

We can, if we wish, put ANY objects together to form a set. {6, Caesar, your great-aunt, Mount Everest} is the set whose elements are the number 6, the man Caesar, your great-aunt, and the mountain Mount Everest. Usually we want to use sets whose elements are ALIKE in some way. For instance, the elements of the set

{2, 4, 6, 8, 10}

are all even numbers. And, the elements of the set

{a, e, i, o, u}

are all vowels in the English alphabet.

Suppose we want to talk about the set whose elements are the numbers 1, 2, 3, 4, and 5. Then, for convenience, let's use some capital letter, perhaps N, to represent this set. Now we may write

N = {1, 2, 3, 4, 5}.

This is read, "N is the set of numbers 1, 2, 3, 4, and 5."

If a set contains 1 quarter, 1 dime, 1 nickel, and 1 penny, we may describe this using T to represent the set, in this way:

T = {quarter, dime, nickel, penny}.

You may choose ANY capital letter to represent a set. If we have the set S = {2, 4, 6, 8}, we can describe this by saying, "S is the set of even numbers from 2 through 8."

Here are a few more examples of sets that will help make the idea clear. On the left we have named them by listing the elements of each set. On the right we have stated how the elements of each set are alike.

Listing of Elements	Word Description
1. V = {a, e, i, o, u}	1. V is the set of vowels in our alphabet.
2. P = {1, 3, 5, 7, 9}	2. P is the set of odd numbers from 1 through 9.
3. E = {Michigan, New York, Ohio, Pennsylvania}	3. E is the set of states in the USA which are touched by Lake Erie.
4. G = {Hawaii}	4. G is the set of states of the USA which are islands.

Exercises 3-2b

List the names of the ELEMENTS of the following sets.

1. The set of all months of the year whose names begin with A. Call the set M. List the set in the following way:
   M = {April, ...}

2. The set of all days of the week whose names begin with S. Call the set D.

3. The set, T, which is made up of all the subjects you are now taking at school.

4. The set, S, of all the states in the United States whose names begin with M.

5. The set, V, of the names of all the varsity sports at your school.

6. The set, R, of all the even numbers starting with 10 and stopping with 24.

7. The set, A, composed of the last four letters of the English alphabet.

8. The set, B, of all the odd numbers larger than 21, but smaller than 35.

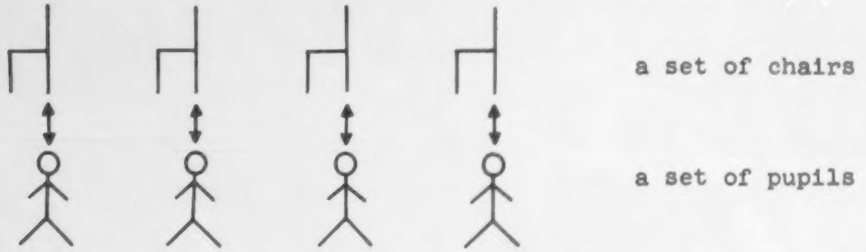
Write a description of each of the following sets from 9 to 16. Keep your description as brief and accurate as you can.

9. C = {Washington D.C., London, Paris}
10. D = {13, 15, 17}
11. W = {penny, nickel, dime, quarter, half-dollar}
12. A = {a, b, c, d, e, f}
13. H = {3, 6, 9, 12, 15, 18, 21}
14. S = {Alaska, Alabama, Arizona, Arkansas}
15. T = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
16. M = {1, 3, 5, 7, 9}

3-3. Counting Numbers.

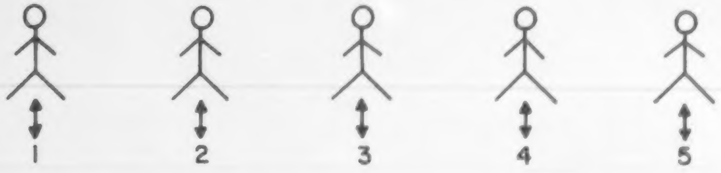
In Chapter 2 you learned that people have needed to use numbers since very early times. These numbers were used to count such things as arrows, sheep, and animal skins. The idea of counting began when people tried to match objects in one set with the objects in another set. Thus, early man tried to match a set of pebbles with his set of sheep. The matching of a given number of sheep with the same number of pebbles is an example of a one-to-one correspondence.

Imagine that each pupil in your room is seated and there are no empty chairs. Each pupil is matched with the chair he sits on. This is an example of a one-to-one correspondence.



In the above example of a one-to-one correspondence we see that there is only one chair for each person and that chair corresponds to only that person. The "pairing" is in both directions.

In mathematics we have a special set of numbers we use in counting. This is the set $\{1, 2, 3, 4, 5, \ldots\}$ (The three dots mean "and so on" as far as we want to go.). We call this set of numbers the counting numbers. To count the pupils in your room, we match each pupil with a counting number. In other words, we make a one-to-one correspondence.



We can now see that there are 5 pupils in the picture above.

The last counting number in the matching tells us "how many" pupils are in the class.

What about zero? Is zero a counting number? Suppose you wanted to count the marbles in a bag. Would you say "Here is marble number zero"? Of course, you would not. But suppose the bag is empty! Then we might say there are "no marbles," or there are "zero marbles" in the bag. But, we DO NOT count them. So let's agree that zero is NOT a counting number.

The first counting number we have, then, is 1. We call the counting numbers and zero the set of whole numbers:

$$\{0, 1, 2, 3, 4, \dots\}$$

Exercises 3-3

1. List the elements of these sets.
   a. The set of continents.
   b. The set of oceans.
   c. The set of the first 10 odd whole numbers.
   d. The set of the first 10 counting numbers.
   e. The set of the first 5 whole numbers.
   f. The set of days of the week.
   g. The set of members of your family.
   h. The set of squares of the first five counting numbers.

2. Is there a mistake in counting the number of x's in this figure?

   x x x x x x x x x x x x x
   1 2 3 4 5 6 7 9 10 11 12 13 14

   If your answer is yes, what is the mistake?

3. Which of the numbers 0, 2, 5, 7, 8, 11, are:
   a. NOT counting numbers between one and ten?
   b. NOT counting numbers between six and eleven?

4. Ed had 16 tickets to sell for a dance. The tickets were numbered in order. The first ticket was marked with the numeral 2. What numeral was on the last ticket?

5. The following shows a one-to-one correspondence between the __________ numbers and the __________ numbers.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & 10 & 12 & \dots \end{array}$$

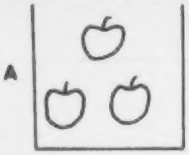
6. A theatre owner wants to know how many people attended his theatre last night. He knows the first ticket was marked 27 and the last ticket was marked 81. How did he figure that 54 people attended? Explain why he was incorrect.

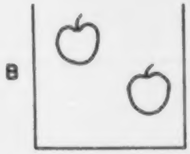
3-4. Properties of Operations.

Everything around us, including people, is described by certain properties. We may describe a person by his age, height, weight, or color of his hair. Sets of numbers are also described by their properties. For instance, even numbers are numbers which are exactly divisible by 2. This is a property of even numbers.

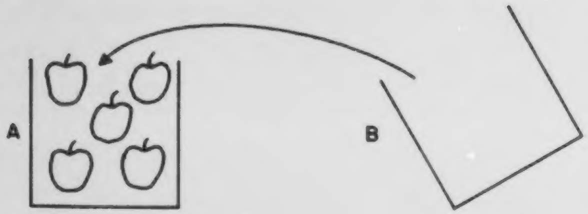
You have studied whole numbers for a long time. You know the operations which we call addition, multiplication, subtraction, and division. Each of these is an operation we can perform with a pair of numbers. These four operations have certain properties which you have used, but perhaps never named. We are going to talk about these special properties and give each a name.

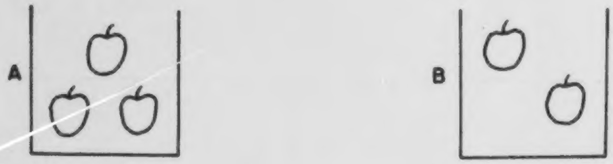
3-5. Commutative Property.



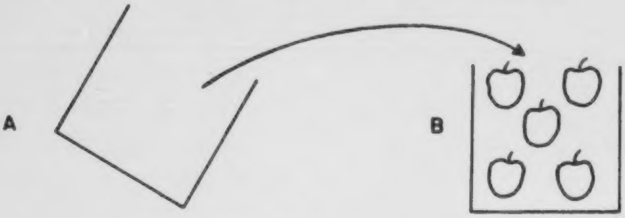


Here are two sets of apples. The A set has three apples. The B set has two apples. We can put the two sets together by dumping the B set into the A set. We now have a new set with 5 elements, which was obtained by adding 3 + 2.





We could also put the two sets together by dumping the A set into the B set. This would be adding 2 + 3. Would we get a different number of apples by this method?



Class Exercises 3-5a

1. Place in the blank spaces the numeral that makes the statement true.

a. ___ + 2 = 2 + 3
b. 76 + ___ = 24 + 76
c. 82 + 63 = 63 + ___
d. 8 × ___ = 3 × 8
e. ___ × 7 = 7 × 6
f. 5 × 2 = ___ × 5
g. 12 + 9 = ___ + 12
h. 6 × 8 = 8 × ___
*i. a + b = b + ___
*j. a × b = ___ × a
*k. ___ + c = c + d
*l. c × d = d × ___

2. Interchange the numbers in each of the following. For example, if we interchange the numbers in "2 + 5" we get "5 + 2." In which ones do the answers stay the same?
   a. $1 + 2 =$
   b. $6 + 8 =$
   c. $7 \times 9 =$
   d. $4 \times 3 =$
   e. $12 \div 3 =$
   f. $9 \div 4 =$
   g. $3 \div 12 =$
   h. $4 \div 9 =$
   i. $5 - 4 =$
   j. $6 - 3 =$

3. Answer the following questions:
   a. When we *add* two numbers, does it make any difference which number we name first? (Does $9 + 5 = 5 + 9$?)
   b. When we *multiply* two numbers, does it make any difference which number we name first? (Does $8 \times 7 = 7 \times 8$?)
   c. When we *subtract* one number from another, does it make any difference which number comes first? (Does $9 - 3 = 3 - 9$?)
   d. When we *divide* one number by another number, does it make any difference which number we take first? (Does $15 \div 5 = 5 \div 15$?)

Suppose you want to perform an operation on a pair of numbers. You want to combine these numbers to get a *certain definite* answer. If it doesn't make any difference which number you start with, the operation is *commutative*. This means that the order in which we write the numerals doesn't change the answer. We say that the operation has the *commutative property*.

Exercises 3-5b

1. Which of the four operations of addition, subtraction, multiplication, and division are commutative? Give examples to support your decision.

2. Which of the following activities are commutative?
   a. To put on a hat and then a coat.
   b. To put on your socks and then your shoes.
   c. To rake the leaves and burn them.
   d. To put on your left shoe and then your right shoe.
   e. To build a house and then move into that house.
   f. To open a book and then read that book.

3. Add. Then use the commutative property to check your addition.

   a. $\begin{array}{r}46\\ \underline{17}\end{array}$ b. $\begin{array}{r}25\\ \underline{32}\end{array}$ c. $\begin{array}{r}22\\ \underline{57}\end{array}$ d. $\begin{array}{r}727\\ \underline{324}\end{array}$ e. $\begin{array}{r}809\\ \underline{672}\end{array}$

4. Multiply. Then use the commutative property to check your multiplication.

   a. $\begin{array}{r}46\\ \underline{17}\end{array}$ b. $\begin{array}{r}25\\ \underline{32}\end{array}$ c. $\begin{array}{r}57\\ \underline{89}\end{array}$ d. $\begin{array}{r}607\\ \underline{302}\end{array}$ e. $\begin{array}{r}809\\ \underline{672}\end{array}$

When you see the symbols ♤ , ♢ , ♡ , ♧ , on cards you probably think of the names of spades, diamonds, hearts, and clubs. These are symbols that you recognize right away. In mathematics you are familiar with many symbols. You know that + means "add", - means "subtract", × means "multiply", ÷ means "divide", and = means "equals". We use symbols in mathematics to simplify our writing of mathematical expressions and statements.

Any symbol can be introduced and used if we all agree upon what we want the symbol to mean. We often need to say that one number is greater than another, or that one number is less than another. Also, we may have to say that one number is not equal to another number. Mathematicians have symbols to indicate these relationships, as follows.

$<$ means "is less than"
$>$ means "is greater than"
$\cdot$ means "is multiplied by"
$\neq$ means "is not equal to"

There is a danger that the letter "x" and the multiplication sign may be mistaken for each other. For this reason we use the raised dot, $\cdot$, to mean "multiplication".

Oral Exercises 3-5c

Read the following mathematical statements:

1. $2 < 6$
2. $3 \cdot 7 = 21$
3. $3 \neq 2$
4. $8 \neq 11$
5. $14 + 15 < 16 + 18$
6. $8 \cdot 25 = 200$
7. $92 > 25$
8. $9 \cdot 8 = 72$
9. $4 \neq 17$
10. $11 > 6$
11. $19 - 17 < 5$
12. $14 \div 7 < 5$
13. $16 > 8 > 2$
14. $3 < 10 < 14$

Exercises 3-5d

Place the symbol $=$, $<$, or $>$ in the place of the question mark to make the statement true.

1. $7 + 4$? $4 + 7$
2. $12 \cdot 5$? $5 \cdot 11$
3. $3 \cdot 12$? $12 \cdot 2$
4. $24 + 3$? $3 + 24$
5. 3 ? 6
6. $8 - 3$? $9 - 3$
7. $19 \cdot 5$? $5 \cdot 19$
8. $35 \div 7$? $49 \div 7$
9. $12 \cdot 8$? $8 \cdot 12$
10. $(3 \cdot 2) + 5$? $5 + (3 \cdot 2)$
11. 16 ? 9 ? 3
12. 0 ? 1 ? 2
13. 10^2 ? 20
14. 2^3 ? 8
15. 4^5 ? 5^4
16. 2^3 ? 3^2

The practice of using a letter to stand for any number is useful in mathematics. Letters are helpful in stating general ideas.

You have shown that the commutative property for addition and multiplication is true for certain pairs of numbers. For example, we know that

$4 + 3 = 3 + 4$ $\qquad$ $3 \cdot 2 = 2 \cdot 3$
$7 + 6 = 6 + 7$ $\qquad$ $9 \cdot 10 = 10 \cdot 9$
$96 + 42 = 42 + 96$ $\qquad$ $11 \cdot 226 = 226 \cdot 11$

It is not possible to list every pair of numbers that we use in addition and multiplication. It would be helpful, then, to state the commutative property so that it includes *all* pairs of counting numbers. Letters will help us do just that.

Suppose we let a and b represent any two whole numbers. They may be the same number or they may be different numbers. The commutative property for addition of whole numbers stated in mathematical language is:

Commutative Property for Addition: $a + b = b + a$.

This means that when we add two whole numbers we may change the order of the numbers *without* changing the answer.

The commutative property for multiplication of whole numbers is:

Commutative Property for Multiplication: $a \cdot b = b \cdot a$.

This means that when we multiply two whole numbers we may change the order of the numbers *without* changing the answer.

Exercises 3-5e

List the set of whole numbers which may be used in place of the letter a to make the statement true.

Example	*Solution*
$3 + a = 3 + 5$	$\{5\}$
$2 \cdot 7 = 7 \cdot a$	$\{2\}$
$6 \cdot 3 > a \cdot 4$	$\{0, 1, 2, 3, 4\}$

1. $4 + a = 6 + 4$
2. $5 \cdot 8 = 8 \cdot a$
3. $2 \cdot a = 2 \cdot 1$
4. $3 \cdot a < 3 \cdot 2$
5. $15 \div 3 > a$
6. $32 + a = 17 + 19$
7. $3 + a < 2 + 7$
8. $7 \cdot 3 > a \cdot 5$
9. $a + 3 = 3 + a$
10. $9 - a > 7$
11. The sum of the number a and 4 is 13.
12. The product of 7 and the number a is the same as the the product of 8 and 7.
13. The product of 9 and 6 is greater than the product of the number a and 6.
14. The sum of 8 and a is the same as the sum of a and 8.

3-6. The Associative Property.

What do we mean by $18 \div 6 \div 3$? We can not be sure. All of our work in division has always been with two numbers. But here we have 3 numbers! The expression $18 \div 6 \div 3$ could mean $(18 \div 6) \div 3$, or $18 \div (6 \div 3)$. If it means $(18 \div 6) \div 3$, the answer is $3 \div 3$ or 1. But if it means $18 \div (6 \div 3)$, the answer is $18 \div 2$ or 9. To avoid this confusion, we need to use parentheses to show how we want these numbers grouped. Without parentheses, expressions like $18 \div 6 \div 3$ have no meaning. You can see that grouping is very important for division. Do you suppose this is true for all the basic operations?

Add $5 + 2 + 8$. Did you do this by adding all 3 numbers at once? You probably thought

"$5 + 2 = 7$ and $7 + 8 = 15$."

Or maybe you thought

"$2 + 8 = 10$ and $5 + 10 = 15$."

Both ways give the same answer. And in both examples we added only two numbers at a time. It is very important to remember that we can add, multiply, divide, and subtract with only two numbers at a time.

We have shown above that

$$\begin{aligned}(5 + 2) + 8 &= 7 + 8 \\ &= 15\end{aligned}$$

and $5 + (2 + 8) = 5 + 10$

$= 15.$

Therefore, $(5 + 2) + 8 = 5 + (2 + 8)$

since both $(5 + 2) + 8$ and $5 + (2 + 8)$ are equal to 15. Then the definition of

$$5 + 2 + 8$$

would be $(5 + 2) + 8$ or $5 + (2 + 8)$ since we get the same sum by either manner of grouping.

We say that the operation of addition is associative.

Oral Exercises 3-6a

Show that the following are true.

Example: $(4 + 3) + 2 = 4 + (3 + 2)$

$7 + 2 = 4 + 5$

$9 = 9$

1. $(4 + 7) + 2 = 4 + (7 + 2)$
2. $(46 + 73) + 98 = 46 + (73 + 98)$
3. $34 + (7 + 9) = (34 + 7) + 9$
4. $21 + (5 + 4) = (21 + 5) + 4$
5. What property did you use in each problem above?

Addition, then, has the associative property. Let's look at a few more examples to help make the idea clear.

$$\begin{aligned}(7 + 9) + 11 &= 7 + (9 + 11)\\ &= 7 + 20\\ &= 27\end{aligned}$$

$$\begin{aligned}(12 + 7) + 33 &= 12 + (7 + 33)\\ &= 12 + 40\\ &= 52\end{aligned}$$

$$\begin{aligned}19 + (11 + 57) &= (19 + 11) + 57\\ &= 30 + 57\\ &= 87\end{aligned}$$

Notice in each example that we did not change the order of the numbers. We just grouped them differently. If we try to add the numbers as they are grouped on the left in the last example, the addition becomes a little more difficult than it is on the right.

You have used the associative property many times whenever you have added two numbers like 32 and 7. Usually, you just add the 7 and 2 and then bring down the 3. But 32 means (30 + 2). Then we can write

$$\begin{aligned} 32 + 7 &= (30 + 2) + 7 \\ &= 30 + (2 + 7) \\ &= 30 + 9 \\ &= 39. \end{aligned}$$

Exercises 3-6b

1. Do the following using the associative property.

Example: $$\begin{aligned} 35 + 3 &= (30 + 5) + 3 \\ &= 30 + (5 + 3) \\ &= 30 + 8 \\ &= 38 \end{aligned}$$

a. 15 + 3
b. 33 + 6
c. 72 + 5
d. 96 + 7
e. 34 + 2

2. Use the associative property to perform the following additions as simply as possible. First, insert the parentheses to show how you are grouping the numbers. Then, give the result of your addition.

Example: 32 + 8 + 14

$$(32 + 8) + 14 = 40 + 14 = 54.$$

a. 51 + 9 + 22
b. 16 + 25 + 25
c. 311 + 89 + 76
d. 15 + 14 + 16
e. 23 + 17 + 18
f. 24 + 6 + 87

You have seen that addition is both commutative and associative. Multiplication is commutative. Do you think multiplication is also associative? Let's try an example and find out.

$$\begin{aligned}(6 \cdot 5) \cdot 9 &= 30 \cdot 9\\ &= 270.\end{aligned}$$

Now, group these same numbers differently.

$$\begin{aligned}6 \cdot (5 \cdot 9) &= 6 \cdot 45\\ &= 270.\end{aligned}$$

The result is the same in both examples. This may help you to see that multiplication is associative.

Exercises 3-6c

1. Show that the following equalities are true.

   Example: $5 \cdot (3 \cdot 6) = (5 \cdot 3) \cdot 6$
   $5 \cdot 18 = 15 \cdot 6$
   $90 = 90$

   a. $7 \cdot (3 \cdot 4) = (7 \cdot 3) \cdot 4$
   b. $(5 \cdot 9) \cdot 2 = 5 \cdot (9 \cdot 2)$
   c. $21 \cdot (3 \cdot 5) = (21 \cdot 3) \cdot 5$
   d. $9 \cdot (2 \cdot 8) = (9 \cdot 2) \cdot 8$

2. Replace the question marks with numerals so that the following statements are true.

   a. $(98 + 74) + 31 = 98 + (? + 31)$
   b. $81 \cdot (76 \cdot 42) = (81 \cdot 76) \cdot ?$
   c. $(79 \cdot 89) \cdot 99 = ? \cdot (89 \cdot 99)$
   d. $6 + (8 + 3) = (? + 8) + 3$

3. a. Does $(10 - 7) - 2$ equal $10 - (7 - 2)$?
   b. Does $18 - (5 - 2)$ equal $(18 - 5) - 2$?
   c. What can you say about the associative property for subtraction?

4. a. Does $(32 \div 4) \div 2$ equal $32 \div (4 \div 2)$?
   b. Does $(60 \div 30) \div 2$ equal $60 \div (30 \div 2)$?
   c. Place parentheses in $75 \div 15 \div 5$ so that the answer will equal 1.
   d. Place parentheses in $75 \div 15 \div 5$ so that it will equal 25.
   e. Place parentheses in $80 \div 20 \div 2$ so that it will equal 8.
   f. Place parentheses in $80 \div 20 \div 2$ so that it will equal 2.
   g. What can you say about the associative property for division?

Sometimes in multiplication and in addition we can make the operation easier by rearranging the numbers. This can be done by the use of the *commutative property*. Then the addition or multiplication can be done by grouping, using the *associative property*.

5. Rewrite these problems using the associative and commutative properties *whenever they make the operation easier*. Use parentheses to show the operation which is done first. Then find the answers.

Example:

$25 + (36 + 75)$

$= 25 + (75 + 36)$ Commutative Property of Addition (We have switched the numerals inside the parentheses.)

$= (25 + 75) + 36$ Associative Property of Addition (We have grouped the numerals differently.)

$= 100 + 36$

$= 136$

Example:

$5 \cdot (19 \cdot 2)$

$= 5 \cdot (2 \cdot 19)$ Commutative Property of Multiplication

$= (5 \cdot 2) \cdot 19$ Associative Property of Multiplication

$= 10 \cdot 19$

$= 190$

a. $(6 + 1) + 9$
b. $2 \cdot (13 \cdot 10)$
c. $(12 \cdot 9) \cdot 10$
d. $7 + (12 + 3)$
e. $4 \cdot (25 \cdot 76)$
f. $340 + (522 + 60)$
g. $(5 \cdot 67) \cdot 2$
h. $180 + (20 + 16)$

In Section 3-5d, we showed how to state the commutative property to include *all* pairs of counting numbers. Perhaps we can do the same thing with the associative property.

Associative Property for Addition: Suppose we let a, b, and c represent any whole numbers. Then the associative property for the addition of whole numbers is

$$(a + b) + c = a + (b + c).$$

The associative property for addition means that if we add three numbers we may group them in either way shown above without changing the final result.

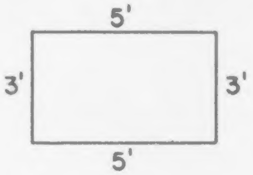
Associative Property for Multiplication: The associative property for multiplication of whole numbers may be written

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

The associative property for multiplication means that if we multiply three numbers we may group them in either way shown above without changing the final result.

3-7. The Distributive Property.

To find the distance around the top of a desk Tom measured the length of each side. His measurements are shown in the drawing. Then he found the distance by adding 5 + 3 + 5 + 3. His answer was 16 feet. Ed said he thought this was all right but it was more work than necessary. He said he would add 5 and 3, then multiply this sum by 2. In mathematical language Ed's idea says



$$\begin{aligned} 2 \cdot (5 + 3) &= 2 \cdot 8 \\ &= 16. \end{aligned}$$

Do you agree that this gives the same answer as 5 + 3 + 5 + 3? Ethel thought it would be better to do the problem a different way. She said,

> "Multiply 5 by 2 because there are 2 sides of 5 feet each.
> Multiply 3 by 2 because there are 2 sides of 3 feet each.
> Add these two products."

Ethel's idea can be written as

$$\begin{aligned}(2 \cdot 5) + (2 \cdot 3) &= 10 + 6 \\ &= 16.\end{aligned}$$

Ed and Ethel were using an idea which is very important in mathematics. It is called the distributive property. Here are several more examples using this property.

Example 1:

John collected money in his homeroom. On Tuesday 7 people gave him 15 cents each, and on Wednesday, 3 people gave him 15 cents each. How much money did he collect? He figured:

the money I collected on Tuesday is $15 \cdot 7$,
the money I collected on Wednesday is $15 \cdot 3$.

Now I'll add these and get the total,

$$\begin{aligned}15 \cdot 7 + 15 \cdot 3 &= 105 + 45 \\ &= 150\end{aligned}$$

so he collected $1.50.

In his mathematics class John had a new idea. He decided to keep a record of how many people paid him each day. He added the numbers, then multiplied by 15 because everyone gave him 15 cents. In symbols his idea looks like this:

$$\begin{aligned}15 \cdot (7 + 3) &= 15 \cdot 10 \\ &= 150\end{aligned}$$

The final answer is the same in both methods. John decided that $15 \cdot 7 + 15 \cdot 3$ and $15 \cdot (7 + 3)$ are names for the same number, 150. This means

$$15 \cdot (7 + 3) = 15 \cdot 7 + 15 \cdot 3$$

He was using the distributive property.

Example 2:

Multiply 14 by 2. You probably do the multiplication like this:

$$\begin{array}{r} 14 \\ \underline{2} \\ 28 \end{array}$$

When you do this, you say $2 \cdot 4 = 8$, and $2 \cdot 1 = 2$, and the answer is 28. Actually "1" stands for "10," and $2 \cdot 10 = 20$. When you write the "2" to the left of 8, you are really adding 20 to 8. This is the short way to do the problem. But you were really using the distributive property.

Since 14 means $10 + 4$, let's try the problem again.

$$\begin{array}{rr} & 10 + 4 \\ & \underline{\times 2} \\ 2 \cdot 4 = & 8 \\ 2 \cdot 10 = & \underline{20} \\ & 20 + 8 \;=\; 28 \end{array}$$

In the example above we are using the fact that

$$\begin{aligned} 2 \cdot 14 &= 2 \cdot (10 + 4) \\ &= 2 \cdot 10 + 2 \cdot 4 \quad \text{by the distributive property} \\ &= 20 + 8 \\ &= 28. \end{aligned}$$

Exercises 3-7a

1. Perform the indicated operations.
   - a. $3 \cdot (9 + 6)$
   - b. $(8 + 7) \cdot 9$
   - c. $(3 \cdot 9) + (3 \cdot 6)$
   - d. $(14 \cdot 3) + (7 \cdot 3)$
   - e. $6 \cdot (7 + 3)$
   - f. $6 \cdot 7 + 6 \cdot 3$
   - g. $7 \cdot 2 + 8 \cdot 2$
   - h. $(7 + 8) \cdot 2$
   - i. $9 \cdot 3 + 6 \cdot 3$
   - j. $6 \cdot (7 + 3 + 2)$
2. Show that the following are true.

   Example: $$\begin{array}{rcl} 3 \cdot (4 + 3) &=& 3 \cdot 4 + 3 \cdot 3 \\ 3 \cdot 7 &=& 12 + 9 \\ 21 &=& 21 \end{array}$$

   If both numbers are the same, the statement is true.
   - a. $4 \cdot (7 + 5) = 4 \cdot 7 + 4 \cdot 5$

b. $6 \cdot 3 + 6 \cdot 4 = 6 \cdot (3 + 4)$
c. $8 \cdot 6 + 7 \cdot 6 = (8 + 7) \cdot 6$
d. $23 \cdot (2 + 3) = 23 \cdot 2 + 23 \cdot 3$
e. $11 \cdot (3 + 4) = 11 \cdot 3 + 11 \cdot 4$
f. $6 \cdot 5 + 6 \cdot 3 = 6 \cdot (5 + 3)$
g. $2 \cdot (12 + 8) = 2 \cdot 12 + 2 \cdot 8$
h. $16 \cdot 3 + 16 \cdot 1 = 16 \cdot (3 + 1)$
i. $(3 \cdot 4) + (3 \cdot 8) = 3 \cdot (4 + 8)$

3. Put a numeral in place of the ? to make the statement true.
a. $3 \cdot (4 + ?) = (3 \cdot 4) + (3 \cdot 3)$
b. $2 \cdot (? + 5) = (2 \cdot 4) + (? \cdot 5)$
c. $13 \cdot (6 + 4) = 13 \cdot ? + 13 \cdot ?$
d. $(2 \cdot 7) + (3 \cdot ?) = (? + ?) \cdot 7$
e. $(? \cdot 4) + (? \cdot 4) = (6 + 7) \cdot 4$

4. Use the distributive property to rewrite each of the following:
Examples: (1) $5 \cdot (2 + 3) = 5 \cdot 2 + 5 \cdot 3$
(2) $(6 \cdot 4) + (6 \cdot 3) = 6 \cdot (4 + 3)$

a. $4 \cdot (2 + 3)$
b. $7 \cdot (4 + 6)$
c. $(9 \cdot 8) + (9 \cdot 2)$
d. $6 \cdot (13 + 27)$
e. $(12 \cdot 5) + (12 \cdot 7)$
f. $(5 \cdot 6) + (5 \cdot 7)$
g. $8 \cdot (14 + 17)$
h. $(6 + 13) \cdot 5$
i. $(5 \cdot 12) + (4 \cdot 12)$
j. $(3 + 5) \cdot 4$

5. Using the distributive property, rewrite the following.
Examples: (1) $10 + 15 = (5 \cdot 2) + (5 \cdot 3)$ or $5 \cdot (2 + 3)$
(2) $15 + 21 = (3 \cdot 5) + (3 \cdot 7)$ or $3 \cdot (5 + 7)$

a. $6 + 4 = (2 \cdot 3) + (2 \cdot 2)$ or ?
b. $12 + 9$
c. $10 + 15$
d. $3 + 6$
e. $12 + 15$
f. $15 + 25$
g. $35 + 40$
h. $30 + 21$
i. $27 + 51$
j. $7 + 28$

When we studied the commutative and associative properties, we found it helpful to write the properties by using letters. In this way we included all the whole numbers and not just a few. It will be helpful to write the distributive property in mathematical language also.

Distributive Property: Suppose a, b, and c are any whole numbers. They may be the same number or they may be different numbers. Then the distributive property can be written as:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

We learned that multiplication is commutative. The distributive property, then, may also be written

$$(b + c) \cdot a = (b \cdot a) + (c \cdot a).$$

The distributive property is the only one you have studied which includes both addition and multiplication. But not all problems which include both addition and multiplication use the distributive property. For example, $(3 \cdot 5) + 4$ does not use this property. $(3 \cdot 5) + 4$ means multiply 3 and 5, then add 4 to the product.

$$\begin{aligned}(3 \cdot 5) + 4 &= 15 + 4 \\ &= 19\end{aligned}$$

However, $3 \cdot (5 + 4)$ does use this property.

$$\begin{aligned}3 \cdot (5 + 4) &= 3 \cdot 5 + 3 \cdot 4 \\ &= 15 + 12 \\ &= 27\end{aligned}$$

Be sure to examine a problem carefully before you try to use the distributive property.

Exercises 3-7b

1. Do the indicated operations. Which ones do not use the distributive property?
    - a. $(3 \cdot 6) + 9$
    - b. $3 \cdot (6 + 9)$
    - c. $5 + (7 \cdot 8)$
    - d. $(5 + 7) \cdot 8$
    - e. $(3 \cdot 4) + (3 \cdot 5)$
    - f. $(3 + 4) + (3 \cdot 5)$
2. In each of the following exercises state which property is used.
    - a. $4 + 3 = 3 + 4$
    - b. $(8 + 2) \cdot 7 = 8 \cdot 7 + 2 \cdot 7$
    - c. $3 + (2 + 4) = (3 + 2) + 4$

d. $(5 \cdot 3) \cdot 8 = 5 \cdot (3 \cdot 8)$
e. $7 \cdot 2 = 2 \cdot 7$
f. $(9 \cdot 12) + (5 \cdot 12) = (9 + 5) \cdot 12$

3. Multiply, then use the commutative property to check your answer.

$$\begin{array}{r} 728 \\ \underline{304} \end{array}$$

SUMMARY OF PROPERTIES OF OPERATIONS

Addition	Multiplication
Commutative Property $a + b = b + a$	Commutative Property $a \cdot b = b \cdot a$
Associative Property $a + (b + c) = (a + b) + c$	Associative Property $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Distributive Property $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	

3-8. The Closure Property.

In Section 3-3 you learned about the set of counting numbers: {1, 2, 3, 4, ...}. We are going to use the set of counting numbers to help us express a new idea. Let us add any two counting numbers. For example: $7 + 9 = 16$. Is the sum a counting number? Is the sum of the following pairs of numbers a counting number?

a. $16 + 23 = ?$
b. $34 + 53 = ?$
c. $150 + 143 = ?$
d. $2 + 2 = ?$

Suppose you pick any two counting numbers. Is the sum always a counting number?

Since the sum of each pair of counting numbers is always a counting number, mathematicians say that the counting numbers are closed under addition. The set of counting numbers, then, has the property we call closure under addition. This means that when we add two counting numbers our answer is always another counting number. It is not necessary to go outside the set of counting numbers for the sum. Hence the set is closed under addition.

Now look at the set of even counting numbers:

$$E = \{2, 4, 6, 8, 10, 12, \ldots\}$$

Add: $2 + 4 = 6$ Is this sum an element of set E?
$4 + 8 = 12$ Is this sum an element of set E?
$10 + 12 = ?$ Is this sum an element of set E?

Suppose you add any two elements of this set. Is the sum always an even number? Is the set of even counting numbers closed under addition? Give a reason for your answer.

Let's think about the set of the first five counting numbers. Call this set A. Then

$A = \{1, 2, 3, 4, 5\}$

Add: $1 + 2 = 3$ Is the sum an element of set A?

$2 + 3 = 5$ Is the sum an element of set A?

$3 + 5 = ?$ Is the sum an element of set A?

The sum of 3 + 5 is 8. Since 8 is not an element of set A, we say 8 is outside our set. Then we can say that set A is not closed under addition.

To test a set of numbers for closure under addition, try to find a pair of elements in the set whose sum is not in the set. If you can find such a pair, the set is not closed under addition. If there is no such pair, the set is closed under addition. This means that a set is closed under addition if the sum of any two elements is an element of the set.

In the same way we can test any set of numbers for closure under multiplication. We look for a pair of elements whose product is not in the set. If there is such a pair, the set is not closed under multiplication. If there is no such pair, the set is closed under multiplication.

Exercises 3-8a

1. Let $Q = \{1, 3, 5, 7, 9, 11, 13, \ldots\}$ be the set of all odd numbers.
   a. Is the sum of any two odd numbers always an odd number?
   b. Is the set of odd numbers closed under addition?
2. Let $M = \{5, 10, 15, 20, 25, \ldots\}$
   a. Give a word description of this set.
   b. Is this set closed under addition?
3. Are the sets in Problems 1 and 2 closed under multiplication?
4. Are the following sets closed under addition?
   a. The set of counting numbers greater than 50.
   b. The set of counting numbers starting with 100 and ending with 999.
   c. The set of counting numbers less than 49.

d. The set of counting numbers whose numerals end in 0.

5. Are the sets of numbers in Problem 4 closed under multiplication?

6. Are all sets of counting numbers which are closed under addition also closed under multiplication? Why?

7. Is the set of counting numbers {1, 2, 3, 4, 5, ...} closed under subtraction? If your answer is no, give an example of a pair of numbers whose difference is not in this set.

8. Are any of the sets of numbers in Problem 4 closed under subtraction?

9. Is the set of counting numbers {1, 2, 3, ...} closed under division? If your answer is no, give an example of a pair of numbers whose quotient is not in the set.

10. Are any of the sets of numbers in Problem 4 closed under division?

Exercises 3-8b

1. Add: a. $\begin{array}{r} 476 \\ 398 \\ 7256 \\ \underline{89} \end{array}$ b. $\begin{array}{r} 403 \\ 213 \\ 414 \\ \underline{898} \end{array}$

2. Subtract: a. $\begin{array}{r} 40302 \\ \underline{20305} \end{array}$ b. $\begin{array}{r} 1777 \\ \underline{598} \end{array}$

3. Multiply: a. $\begin{array}{r} 9816 \\ \underline{8} \end{array}$ b. $\begin{array}{r} 50106 \\ \underline{9} \end{array}$ c. $\begin{array}{r} 357082 \\ \underline{7} \end{array}$

4. Write in words: 2,070,351

5. If 8 oranges cost 48 cents, what is the cost of a dozen oranges?

6. Explain the meaning of the following symbols in words and illustrate each with an example.
   a. $>$
   b. $=$
   c. $<$
   d. $\neq$
   e. $\cdot$

3-9. Inverse Operations.

Listed below are four pairs of operations. How is one operation related to the other in each of these pairs?

a.	Open the door	Close the door
b.	Turn on the light	Turn off the light
c.	Add 6	Subtract 6
d.	Put on your coat	Take off your coat

You may have noticed that one operation will undo the other operation. In mathematics we say each operation is the *inverse* of the other. *Opening* the door is the inverse of *closing* the door. *Closing* the door is also the inverse of *opening* the door. Subtracting 6 is the inverse of adding 6. Can you think of other pairs of inverse operations?

You have used the idea of an inverse operation when you used addition in checking subtraction.

For example:

$$\begin{array}{r} 203 \\ \underline{-\ 96} \\ 107 \end{array} \qquad \text{Check: } \begin{array}{r} 107 \\ \underline{+\ 96} \\ 203 \end{array}$$

Adding 96 will undo the operation of subtracting 96. You should end up with the same number you started with.

$$(203 - 96) + 96 = 203$$

Oral Exercises 3-9a

Some of the words and phrases below have inverse operations and some do not. Find those which do, and give the inverse.

1. Picking up the pencil. ("*Not* picking up the pencil" is not an inverse operation. "*Not* picking up the pencil" does not undo the operation of picking up the pencil.) The inverse of "picking up the pencil" is "laying down the pencil."

2. Put on your hat.
3. Get into a car.
4. Extend your arm.
5. Multiplication.
6. Build.
7. Smell the rose.
8. Step forward.
9. Read a book.
10. Addition.
11. Division.
12. Subtraction.
13. Look at the stars.
14. Talk.
15. Take a tire off a car.

Suppose the athletic fund in your school had \$1800 in the bank. After the last football game, \$300 more was deposited in the bank. The fund then had \$1800 + \$300 or \$2100 in it. But the team needed new uniforms which cost \$300, so \$300 was taken out of the fund to pay for the uniforms. The amount left in the fund was \$2100 - \$300 or \$1800. These operations undo each other.

We can express this idea about addition and subtraction in a more general way by using letters. Suppose we let the number of dollars in the athletic fund before the game be represented by x. If we let b represent the amount we deposited, then

$$x + b = a$$

where a represents the number of dollars we now have in the bank. How shall we undo this operation of adding b? The inverse of adding b is subtracting b from the total amount in the bank. This should give us the number of dollars we started with. Let's express this in symbols:

$$x = a - b.$$

Examples: Hence, if we have

$x + b = a$ then $x = a - b$
$x + 10 = 17$ then $x = 17 - 10$ and $x = 7$
$x + 4 = 12$ then $x = 12 - 4$ and $x = 8$
$x + 20 = 23$ then $x = 23 - 20$ and $x = ?$

Use other whole numbers for a and b and find x as was done in the examples above.

These examples seem to show us that

$$x + b = a \text{ means the same as } x = a - b.$$

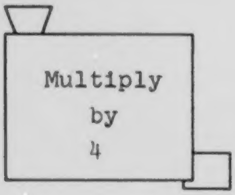
Notice that if a and b are *whole numbers*, and if $a > b$ (a is larger than b) or $a = b$, we can find a *whole number* x so that $b + x = a$.

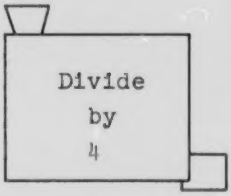
You may have heard it said that division is the inverse of multiplication. Let us try to see what this means. Look at the following statements:

$$\begin{tikzcd} 3 \cdot 4 = 12\,, \arrow[r, leftrightarrow, bend left] & 12 \div 4 = 3\,; \end{tikzcd}$$

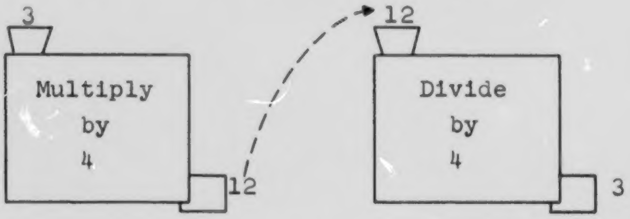
$$\begin{tikzcd} 5 \cdot 4 = 20\,, \arrow[r, leftrightarrow, bend left] & 20 \div 4 = 5\,. \end{tikzcd}$$

Suppose we look at these problems in another way. We have two machines. One machine multiplies numbers by 4; the other machine divides numbers by 4:





We now hook the machines together so that the number that comes out of the second machine is fed into the second.



We see that the number that comes out of the second machine is the same as the number fed into the first machine. The operation performed by the second machine "undoes" the operation of the first machine and gives us back the number we started with. We say that the *operation of dividing by 4 is the inverse of the operation of multiplying by 4*. In the same way the operation of dividing by 29 is the inverse of multiplying by 29.

This is what we mean by saying that division is the inverse operation of multiplication.

This statement can be written for any counting number by using letters. If a, b, and x represent counting numbers then

$$b \cdot x = a \text{ means the same as } x = a \div b.$$

Division by b is the inverse of multiplication by b. And, Multiplication by b is the inverse of division by b.

Oral Exercises 3-9b

Find a whole number which can be used for x to make each of the following true. If there is no whole number that can be used for x, your answer is "none." (Remember that the set of whole numbers is $\{0, 1, 2, 3, \ldots\}$.)

1. $9 + x = 14$
2. $x + 9 = 14$
3. $x + 1 = 2$
4. $4 + x = 11$
5. $10 + x = 7$
6. $5 + x = 5$
7. $10 = x + 2$
8. $x = 9 - 5$
9. $x = 11 - 8$
10. $8 + x = 11$
11. $6 + x = 3$
12. $x = 13 - 6$
13. $3 + x = x + 3$
14. $3 \cdot x = 12$
15. $4 \cdot x = 20$
16. $x = 20 \div 4$
17. $2 \cdot x = 18$
18. $x = 18 \div 2$
19. $5 \cdot x = 30$
20. $2 \cdot x = 0$
21. $x = 0 \div 2$
22. $9 \cdot x = 0$
23. $x = 0 - 9$
24. $3 \cdot x = 3$
25. $x = 3 \div 3$
26. $11 \cdot x = 11$

Exercises 3-9c

1. Perform the indicated operation and check by the inverse operation. Subtract in (a) through (f).

a. $\begin{array}{r} 89231 \\ \underline{42760} \end{array}$

b. $\begin{array}{r} \$805.06 \\ \underline{\$297.96} \end{array}$

c. $\begin{array}{r} 803 \text{ ft.} \\ \underline{297 \text{ ft.}} \end{array}$

d. $\begin{array}{r} \$4302.14 \\ \underline{\$2889.36} \end{array}$

e. $\begin{array}{r} \$8000.02 \\ \underline{\$6898.98} \end{array}$

f. $\begin{array}{r} \$10040.50 \\ \underline{8967.83} \end{array}$

g. $29\overline{)25404}$

h. $38\overline{)37506}$

i. $27\overline{)21546}$

j. $19\overline{)13243}$

2. Find a whole number which can be used for x to make each of the following true. If there is no whole number that can be used for x, write "none".

a. $x + 13 = 25$

b. $27 + x = 58$

c. $14 \cdot x = 42$

d. $17 \cdot x = 85$

e. $940 = x + 352$

f. $219 + x = 682$

g. $327 \cdot x = 981$

h. $215 \cdot x = 1290$

i. $32 + x = 25$

j. $98 \cdot x = 304$

3. a. One bookcase will hold 128 books. A second holds 109 books. How many more books does the first case hold than the second?

b. A theatre sold 4789 tickets in July and 6781 tickets in August. How many more tickets were sold in August than in July?

c. One building has 900 windows and another building has 811 windows. How many more windows does the first building have than the second?

d. The population of a town was 19,891. Five years later the population was 39,110. What was the increase in population for the 5 years?

e. If one truck can carry 2099 boxes, how many boxes can 79 of these trucks carry?

f. How many racks are needed to hold 208 folding chairs if each rack holds 16 chairs?

g. A box of 288 pieces of candy was bought for a party. There were 48 children at the party. How many pieces of candy did each child have?

h. A Girl Scout troup has 29 members. The troup sold 580 boxes of cookies. Each girl sold the same number of boxes. How many boxes of cookies did each girl sell?

4. Perform the following operations.

a. Add 16 and 17. Subtract 12 from this sum.

b. Subtract 24 from 89. Add 19 to this number.

c. Multiply 27 by 34. Divide the product by 9.

d. Find the sum of 9, 9, and 9. Subtract $6 \cdot 4$ from your answer.

e. Divide 308 by 28. Multiply the answer by 5. Then subtract 9 from this product.

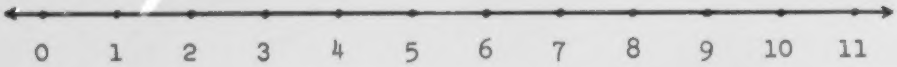
f. Find the difference between 47 and 38. Divide this difference by 3. Now add 17.

g. Divide 272 by 16. Multiply the quotient by 12. Now subtract 100 from the product.

h. Multiply 12 by 13, then add 39.

3-10. Betweenness and the Number Line.

Let us make a drawing that shows how numbers are related. First draw a line with a ruler. Pick a point on this line and label the point with the symbol 0. Use the marks on the ruler to locate points equally spaced along the line as shown below.



Then label each mark on the line with the counting numbers. Label the first mark to the right of zero with a one (1), the second mark with two (2), and so on as shown. This line is called the Number Line. We can think of this line as extending as far as we please so that every whole number can be located on the line.

Is 3 greater than or less than 5? Is the point 3 to the right or to the left of the point 5? Compare the numbers 3 and 2. Which is larger? Compare the points 3 and 2. Which is to the right of the other? If a number is less than another number, how are the corresponding points on the number line located? This table shows these relations.

Relations between numbers	Relations between points
$2 < 3$ (is less than)	2 is to the left of 3
$5 > 3$ (is greater than)	5 is to the right of 3

The compound sentence

$$2 < 3 \quad \text{and} \quad 3 < 5$$

is often abbreviated like this:

$$2 < 3 < 5$$

It tells us that 3 is between 2 and 5 and that 3 is greater than 2 but less than 5.

How many whole numbers are there between 6 and 11? If we look at the number line, we can count them. We see four of them--7, 8, 9, and 10. The number line can thus be used to show how numbers are related.

Exercises 3-10

1. How many whole numbers are there between:
   a. 7 and 25
   b. 3 and 25
   c. 20 and 25
   d. 17 and 25
   e. 25 and 25
   f. 28 and 25
   g. 26 and 25
   h. 114 and 25

2. What is the whole number midway between:
    a. 7 and 13
    b. 9 and 13
    c. 20 and 28
    d. 10 and 50
    e. 17 and 19
    f. 17 and 27
    g. 12 and 20
    h. 12 and 6
3. Which of the following pairs of whole numbers have a whole number midway between them?
    a. 6, 8
    b. 6, 10
    c. 8, 18
    d. 8, 13
    e. 7, 12
    f. 26, 33
    g. 9, 17
    h. 19, 36
    *i. a, b if a and b are even whole numbers.
    *j. a, b if a and b are odd whole numbers.
    *k. a, b if a is odd and b is even.

*4. The whole numbers a, b, and c are located on the number line so that b is between a and c, and $c > b$.
    a. Is $c > a$? Explain with a number line.
    b. Is $b > a$? Explain with a number line.
    c. Is $b < c$? Explain with words.

3-11. The Number One.

The number one is a special number in several ways. It is the smallest of our counting numbers since zero is not a counting number. We can build all of our counting numbers by beginning with 1 and adding 1's. For example, to obtain the number 5, we begin with 1 and repeat the addition of 1: $1 + 1 = 2$, $2 + 1 = 3$, $3 + 1 = 4$, $4 + 1 = 5$. If we choose any counting number such as 78, we get the next counting number by adding 1: $78 + 1 = 79$. One is called the building block of our number system. We can find other special properties of one by working the problems below.

1. a. Multiply: $1 \times 17 =$ ____, $1 \times 39 =$ ____, $1 \times 2001 =$ ____
    b. Multiply: $17 \times 1 =$ ____, $39 \times 1 =$ ____, $2001 \times 1 =$ ____
    c. What is the product when you multiply a number by 1?
    d. What is the product when you multiply 1 by a number?

This shows how the product of one and any number is the original number. Because of this special property of one, the number 1 is called the *identity element for multiplication*. The number one, when used as a multiplier, makes the product *identical* with the multiplicand.

The Multiplication Property of 1: If a represents any counting number, then

$$1 \cdot a = a.$$

2. a. Divide: 13 ÷ 1 = ___, 58 ÷ 1 = ___, 596 ÷ 1 = ___.
   b. Write these quotients as fractions: 1 ÷ 13 = ___, 1 ÷ 58 = ___, 1 ÷ 596 = ___.
   c. When a number is divided by 1, is the quotient a counting number?
   d. When 1 is divided by a counting number, is the quotient a counting number?

A number multiplied by 1 is the same as 1 multiplied by the number. This is true because of the commutative property of multiplication. For example, $3 \cdot 1 = 1 \cdot 3$. Since division is the inverse operation of multiplication, is the number one also special in division? You may have noticed in Problem 2 that when we divide any counting number by one, we obtain the same counting number. But if we divide 1 by any counting number, other than 1, the quotient is not a counting number. One is *not* the identity element for division.

3. a. Divide: 5 ÷ 5 = ___, 27 ÷ 27 = ___, 55 ÷ 55 = ___.
   b. What is the quotient when any counting number is divided by itself?

The simplest numeral for one is "1". But this special number one can also be written with different numerals like "$\frac{2}{2}$", or "$\frac{5}{5}$", as shown in Problem 3a above. These fractions are numerals for one because the quotient of a number (except zero) divided by itself is always one. We shall use numerals for one such as "$\frac{3}{3}$", or "$\frac{10}{10}$", to solve fraction problems later on in our work.

We learned that

$$\begin{aligned} & 10^2 \text{ means } 10 \cdot 10 \text{ or } 100 \\ & 10^3 \text{ means } 10 \cdot 10 \cdot 10 \text{ or } 1000 \\ \text{and } & 10^6 \text{ means } 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \text{ or } 1{,}000{,}000 \end{aligned}$$

The 2, 3, and 6 are called exponents. These three exponents are small numbers, but the numbers represented by 10^2, 10^3, and 10^6 are very large. What happens when we use powers of 1 instead of 10? How much is

$$1^2 \text{ ?} \qquad 1^3 \text{ ?} \qquad 1^6 \text{ ?}$$

Can you compute 1^{200} mentally?

When we write 1^4 or 1^{200}, we are just writing a different name for 1, since 1^{200} is really only 1. The number represented by the completion of these operations is 1, for example, $1^4 = 1 \cdot 1 \cdot 1 \cdot 1 = 1$. Can you think of other combinations of number symbols which represent 1? What number is represented by $5 - 4$? X - IX? If we let c represent any counting number, we can express the special multiplication and division properties of the number 1 with these mathematical sentences. Can you translate them into words?

4. a. $c \cdot 1 = c$
   b. $c \div 1 = c$
   c. $c \div c = 1$
   d. $1 \div c = \frac{1}{c}$
   e. $1^c = 1$

Exercises 3-11

1. From the following symbols, select those that represent the number 1.
   a. I
   b. $\frac{4}{4}$
   c. $5 - 4$
   d. $1 - 0$
   e. $1 + 0$
   f. $1 \cdot 2$
   g. $\frac{6}{4}$
   h. $\frac{4}{1}$
   i. 1
   j. $1 \cdot 0$
   k. $\frac{200}{200}$
   l. 1^{10}
   m. $\frac{1}{2} + \frac{1}{2}$
   n. $1 \cdot 100$
   o. $\frac{8 - 1}{12 - 5}$
   p. $\frac{2 - 1}{1}$

2. Copy and fill in the blanks.
   a. $100 \cdot 1 =$ ____
   b. $10 \cdot 1 \cdot 1 \cdot 1 =$ ____
   c. $\frac{14}{1} =$ ____
   d. $1 \cdot \frac{2}{3} =$ ____
   e. $0 \cdot 1 =$ ____
   f. $1 \cdot 0 =$ ____

*3. Robert said, "The counting numbers are not closed under the subtraction of ones but they are closed under the addition of ones." Show by an example what Robert meant.

4. Perform the indicated operations.

   a. $(4 - 3)\overline{)876429}$

   b. $1\overline{)976538}$

   c. $897638 \cdot (5 - 4)$

   d. $896758 \cdot \frac{4}{4}$

   e. $3479 \cdot 1^{110}$

   f. $97 \cdot x^6$ (if x is 1)

   g. $1^7 \cdot (489 \div 489)$

   h. $(\frac{8}{8} \cdot 1^5) \div 1^4$

3-12. The Number Zero.

Another special number is zero. Sometimes you will hear it called by other names such as "naught" or "oh." When you answer a telephone a voice may say, "Is this 'one eight oh three'?" Of course, "oh" does not refer to the letter "o" and we all understand that the person means "one eight zero three."

The set of whole numbers is the set of counting numbers together with the number zero. In mathematics we try to say exactly what we mean, and we often use words in a special way to make our meaning clear. For this reason we have decided to use the phrase "counting number" to mean any of the numbers 1, 2, 3, ... If we wish to include zero, we say "whole numbers."

$$\underbrace{0, \overbrace{1, 2, 3, 4, 5, \ldots}^{\text{Counting numbers}}}_{\text{Whole numbers}}$$

You see that the set of counting numbers is contained in the set of whole numbers. Many of the interesting properties of counting numbers apply also to zero, and therefore to all the whole numbers.

If you withdraw all your money from the bank, you can express your bank balance with this special number zero. If you have answered no questions correctly, the number you have correct is zero. If there are no chalkboard erasers in the classroom, the number of erasers is zero.

On a very cold morning Paul was asked the temperature. After looking at the thermometer he replied, "zero." Did he mean there was "not any"? Did he mean "nothing"? No, he meant the top of the mercury in the thermometer was at a certain point on the scale labeled zero.

When the thermometer indicates zero, it does not mean that the temperature is "nothing". It means we have a certain temperature which is called zero. In the same way zero on the number line shows how zero is related to the counting numbers. It is as real and specific as 7 or any other number.

We shall now study some of the special properties of zero. What is the sum of a whole number and zero? How does the sum compare with the original number? $4 + 0 = 4$. What is the sum of zero and a whole number? We might express this fact in symbols.

The Addition Property of 0: If c represents any whole number, then

$$c + 0 = c.$$

Or we might express the fact by saying that zero is the identity element for addition.

What is the difference between a whole number and itself, for example, $4 - 4 = ?$ In this subtraction operation, we do not get a counting number. The difference is the special number zero. To put the idea in mathematical language, we would say that the set of counting numbers is not closed under subtraction.

Let us look at the special number zero under the operation of multiplication. What could $3 \cdot 0$ mean? It could mean the number of chairs in 3 rooms if each room contained zero chairs. Thus, any number of rooms containing zero chairs each would have a total of zero chairs. We might express this idea in symbols, $c \cdot 0 = 0$, where c is any counting number.

What could $0 \cdot 3$ mean? If there are no rooms, and 3 chairs in each room, how many chairs are there altogether? It sounds strange, but it makes sense to say that there are no chairs. Let us check by looking at the problem in another way. We know by the commutative property for multiplication that $3 \cdot 0 = 0 \cdot 3$. If a represents any whole number, we may express this by writing $0 \cdot a = 0$. If a is zero, we must have $0 \cdot 0 = 0$.

What is the product of two or more whole numbers if zero is one of the factors? For example, $4 \cdot 5 \cdot 0 = ?$ By the associative property

$$\begin{aligned} 4 \cdot 5 \cdot 0 &= (4 \cdot 5) \cdot 0 \\ &= 20 \cdot 0 \\ &= 0 \end{aligned}$$

In mathematics you will use this fact many times:

> If one of the factors is zero then the product of two or more whole numbers is zero.

Be sure to remember this. There is another very important fact expressed in the example $4 \cdot 5 \cdot 0$. You will use this idea often, too.

> If the product of two or more whole numbers is zero, then one or more of the factors must be zero.

Let us see how zero behaves in division.

What could zero divided by 3 mean? If there is the same number of chairs in each of 3 rooms and there are 0 chairs in all, how many chairs are there in each room? With this meaning, $0 \div 3$ should be 0. Let us check. Since $0 \div 3$ means the number x for which $3 \cdot x = 0$, try replacing "x" by "0" in this number sentence. Do you obtain a true statement? Do you obtain a true statement if x is any other number?

Don't forget this important result: when zero is divided by a counting number, the quotient is always zero and never a counting number. For example, $\frac{0}{7} = 0$.

Now that we have learned how to divide 0 by a counting number, let us see what happens when we try to divide by 0. What is $7 \div 0$? Suppose that

$$7 \div 0 = x.$$

This statement means the same as

$$7 = 0 \cdot x.$$

But if we multiply any number by 0, we obtain 0, not 7. Therefore, we see that there is no such number x. We see that $7 \div 0$ is meaningless. We cannot divide 7 by 0. If r is a counting number, is there any such number as $r \div 0$? Imitate our reasoning with the example $3 \div 0$. As you see,

we cannot divide a counting number by 0.

Let us try dividing 0 by 0. What is $0 \div 0$? Suppose that

$$0 \div 0 = x.$$

This means the same as

$$0 \cdot x = 0.$$

What number must x be? Is this true if x is 1? Is it true if x is 0? Or if x is 99? As you see, x can be any number whatsoever. We say that $0 \div 0$ is meaningless because it can be any number, and is not one definite number.

We can conclude, then, that division by 0 is meaningless. We cannot divide any number by 0.

Some properties of the special numbers zero and one are in this table. State them in words if a and b represent any whole numbers and c represents any counting number.

Properties of 1

a. $1 \cdot a = a$
b. $a \div 1 = a$
c. $\frac{c}{c} = c \div c = 1$
d. $1^c = 1$

Properties of 0

a. $a + 0 = a$
b. $a - 0 = a$
c. $a - a = 0$
d. $0^c = 0$
e. $a \cdot 0 = 0$
f. $0 \div c = 0$
g. $a \div 0$ is meaningless

Exercises 3-12

1. Select the symbols that represent zero.

a. $1 + 0$
b. 0
c. $\frac{4}{0}$
d. $\frac{0}{4}$
e. $5 - 4$
f. $7 - 7$
g. $\frac{a}{0}$
h. $0 + 0$
i. $1 - 1$
j. $100 - 100$
k. $0 \cdot 4$
l. $4 \cdot 0$

m. $0 \cdot 0$

n. $\frac{0}{10}$

o. $\frac{4 - 4}{2}$

p. $\frac{4}{2 - 2}$

q. $\frac{1}{2} - \frac{1}{2}$

r. $1^4 \cdot 25$

s. $12 \cdot 0$

t. $0 + 12$

u. $2(4 + 6 + 0)$

v. $(2 \cdot 4) - 0$

w. $\frac{4}{4}$

x. $\frac{36}{9} - \frac{36}{12}$

2. Perform the indicated operations, if possible.

a. $76 \cdot 49$

b. $78 \cdot 946$

c. $8984 \div 62$

d. $9484 \div 62$

e. $87 \times \$419$

f. $69 \times \$876$

g. $\$989.26\ (2 - 2)$

h. $1 \cdot \$846.25$

i. $5 \times \$14.13$

j. $679 \cdot \frac{4}{4}$

k. $379\ (146.8 - 145.8)$

l. $(34.6 - 33.6)\ 897$

m. $\$397.16 \div (4 - 3)$

n. $\$897.40 \div (3 - 3)$

o. $(480 + 24) \div 20$

p. $\$1846 \div (\frac{1}{2} + \frac{1}{2})$

q. $487.97 \cdot \frac{4}{4} \cdot 0$

r. $49 \cdot 0 \cdot 47 \cdot 97$

s. $\$97.86 \times 0 \times 0$

t. $(9 - 9) \cdot \frac{7 + 2}{8 + 1}$

u. $976 \cdot 1^6$

v. $1^{12} \cdot \$97.46$

*3. Can you find an error in any of the following statements? a and b are whole numbers.

a. $4 \cdot 0 = 0$

b. $0 \cdot 4 = 0$

c. $2 \cdot 1 = 2$

d. $1 \cdot 2 = 2$

e. If $a \cdot b = 0$, then a or $b = 0$

f. If $a \cdot b = 1$, then a or $b = 1$

g. If $a \cdot b = 2$, then a or $b = 2$

h. If $a \cdot b = 3$, then a or $b = 3$

i. If $a \cdot b = C$, then a or $b = C$

4. BRAINBUSTER. If I add 1000 to a certain whole number the result is actually more than if I multiplied that number by 1000. What is the number?

3-13. Summary.

1. The set of numerals {1, 2, 3, 4, 5, ...} is the set of symbols for the counting numbers.

2. The set of numerals $\{0, 1, 2, 3, 4, 5, \ldots\}$ is the set of symbols for the whole numbers.
3. COMMUTATIVE PROPERTY FOR ADDITION: $a + b = b + a$, where a and b represent any whole numbers.
4. COMMUTATIVE PROPERTY FOR MULTIPLICATION: $a \cdot b = b \cdot a$, where a and b represent any whole numbers.
5. ASSOCIATIVE PROPERTY FOR ADDITION: $a + (b + c) = (a + b) + c$ where a, b, c represent any whole numbers.
6. ASSOCIATIVE PROPERTY FOR MULTIPLICATION: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ where a, b, c represent any whole numbers.
7. DISTRIBUTIVE PROPERTY: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ where a, b, c are any whole numbers.
8. New symbols: $\{$set of elements$\}$; $>$ is greater than; $<$ is less than; $\neq$ is not equal to; the raised dot $\cdot$ means times.
9. Set and closure. A set is closed under an operation if the combination of any two elements of the set gives an element of the set. The set of counting numbers is closed under addition and multiplication but not under division or subtraction.
10. Inverse operations. Subtraction is the inverse of addition, but subtraction is not always possible in the set of whole numbers. Division is the inverse of multiplication, but division is not always possible in the set of whole numbers. That is, division of one whole number by another whole number does not always give a whole number.
11. The number line and betweenness. Each whole number is associated with a point on the number line. There is not always a whole number between two whole numbers.
12. Special numbers: 0 and 1. Zero is the identity for addition; 1 is the identity for multiplication; division by 0 is not possible.

3-14. Chapter Review.

1. $(7 \cdot 3) + (3 \cdot 13) = (7 \cdot 3) + (13 \cdot 3)$. In the example shown, what property of whole numbers is illustrated?
2. In the set of counting numbers the identity element for multiplication is ________.
3. Use the distributive property and rewrite:

$$(2 \cdot 13) + (5 \cdot 13)$$

4. Use the associative property of addition so that the sum can be found easily:

$$136 + 25 + 75$$

5. Check by the inverse operation to see if $715 \div 11 = 65$
6. The number of counting numbers between 6 and 47 is ________.
7. How does the set of counting numbers differ from the set of whole numbers?
8. Is this statement true? "I can show that the set of whole numbers is closed with respect to subtraction if I can find one example such as $12 - 8 = 4$ to illustrate this."
9. The value of 1^{12} is ________.
10. How many counting numbers are there between 5 and 6?
11. Place parentheses in $100 \div 20 \div 5$ so that it will equal 25.
12. The distributive property involves two operations: ________ and ________.
13. Find a whole number or whole numbers which may be used in place of b to make the statements true.
    a. $6 + b = 6 + 5$
    b. $b \cdot 7 = 7 \cdot b$
    c. $b + 1 > 3$
    d. $(8 + 3) + 2 = 8 + (3 + b)$
14. The identity element for addition of whole numbers is ________.
15. Is the set $M = \{3, 6, 9, 12\}$ closed under addition? Support your answer with examples.
16. a. The inverse operation of division is ________.
    b. The inverse operation of subtraction is ________.
17. $5 - 4$, $7 \div 7$, and $1 - 0$ are different symbols all of which represent the number ________.
18. Write each of the following in words.
    a. $7 > 2$ b. $15 < 33$ c. $4 < 6 < 10$

3-15. Cumulative Review.

Exercises 3-15

1. $(122)_{three} = (\ \ )_{ten} = (\ \ )_{five}$
2. A girl went to the pantry with only a 5-cup and a 3-cup container to get 4 cups of flour. Can this be done if nothing but the flour container is used in addition to the two containers? If yes, how?
3. Write MCXI in Hindu-Arabic numerals.
4. What is another way of writing 4^5 ?
5. The base of the numeral system that has the easiest multiplication facts to learn is ____________.
6. $(2010)_{three}$ written in decimal notation is:

   $(2 \times \_\_\_) + (0 \times \_\_\_) + (1 \times \_\_\_) + (0 \times 1) = \_\_\_\_\_$
7. If we were to use base 21 numeration we would have ______ different symbols.
8. In base six write the next five numerals after 55_{six}.
9. $(101101)_{two} = (\ ?\ )_{ten}$
10. The set of whole numbers differs in what (if any) way from the set of counting numbers?
11. $(6 \cdot 2) + (2 \cdot 9) = (6 \cdot 2) + (9 \cdot 2)$. In the example shown, what property of whole numbers is illustrated?
12. In the set of counting numbers, one is the identity element for ______________.
13. Use the distributive property and rewrite: $(3 \cdot 5) + (2 \cdot 5)$
14. Use the associative property of addition so that the sum can be found easily. $125 + 75 + 36$
15. Check by the inverse operation to see if $375 \div 3 = 125$.
16. The number of counting numbers between 8 and 46 is ______.
17. The inverse operation for addition is ____________.
18. The inverse operation for multiplication is ____________.
19. Write each of the following in words.

    a. $11 > 8$ b. $6 \not< 10$ c. $2 < 4$

Chapter 4

NON-METRIC GEOMETRY I

We are living at a time when people are interested in travel in space. We are sending rockets into space and some day hope to land a man on the moon. To a mathematician, the word "space" has a broader meaning than the idea of space used in space travel. The mathematician sees many more applications of the space idea. The study of space and location in space is part of mathematics. It is called geometry. In this chapter, we shall study some of the ideas of geometry which will help us to understand points, lines, planes, and space. This chapter is called "non-metric geometry" because we do not use the idea of distance or measurement. You might also call it "no-measurement geometry."

In Chapter 3, you learned a little about sets. You noticed that some ideas are easier to explain if set language is used. In mathematics, the ideas about sets are useful in both arithmetic and geometry. The language of sets may be new to you. You will find, however, that this language helps you to think, to talk, and to read more accurately.

For over 4,000 years men have studied geometry in some form in order to obtain a better understanding of the world in which they lived. Over 2,000 years ago a Greek mathematician, named Euclid, collected and arranged the facts then known about geometry. Even today we call this part of mathematics Euclidean Geometry. Our geometry is the same, but some of our words and some of our ways of looking at things are quite different.

4-1. Points, Lines, and Space.

Points

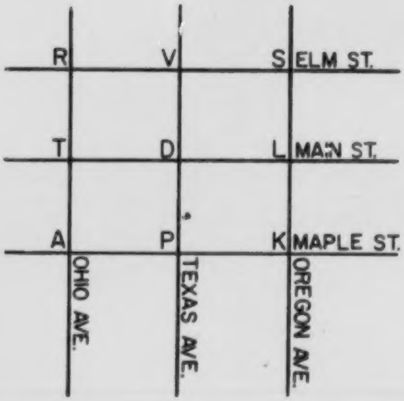
Let us begin our study of geometry with the idea of point. We can think of a point in geometry as a definite location in space. Familiar examples of points are suggested by:

1. the North Pole
2. a chalk dot on a chalkboard.
3. a button on a man's shirt
4. a distant star

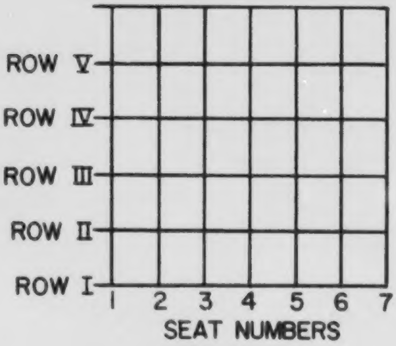
Since we shall be talking about points it is convenient to have a system of naming points. "Mabel Smith" or "Elmer Jones" could be used as names for points. However, we find it simpler to name points by using capital letters. For example, the points represented by the dots below are named by capital letters.

Exercises 4-1a

1. Find objects which suggest points:
   a. in the classroom.
   b. in the home.
   c. in the street.
2. On the map at the right, name the point described where:
   a. Main Street meets Ohio Avenue.
   b. Texas Avenue meets Maple Street.
   c. Elm Street meets Oregon Avenue.
   d. Texas Avenue meets Main Street.



3. A section of a school auditorium consists of rows marked I, II, III, IV, V. In each row, the seats are numbered 1, 2, 3, 4, 5, 6, 7, as shown.

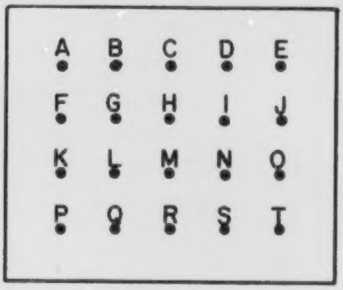


Copy this diagram on your paper. Represent the following locations by dots. Name the dots (points) by using capital letters as indicated below:

a. Row I seat 4 (A)
b. Row III seat 7 (B)
c. Row V seat 2 (C)
d. Row IV seat 5 (D)
e. Row II seat 3 (E)
f. Row I seat 6 (F)

4. The diagram at the right represents a key rack. State the names of the key hooks described below.

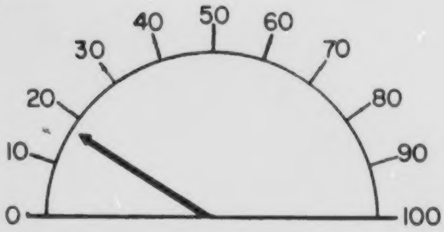
a. Second row from the top, third hook from the left.
b. Top row, hook farthest to the right.
c. Top row, second hook from the left.
d. Third row from the top, second hook from the right.



5. Make a diagram of a baseball field. Locate and name with capital letters the following points:

A--Home plate
B--First base
C--Second base
D--Third base
E--Shortstop
F--Pitcher's box
G--Right field corner
H--Center field corner
J--Left field corner

6. The diagram at the right represents a speedometer of a car. Make a copy of this diagram. On your diagram locate and name with capital letters the points which represent each of the following speeds.



a. 30 miles per hour
b. 42 miles per hour
c. 55 miles per hour
d. 61 miles per hour
e. 18 miles per hour
f. 75 miles per hour

Lines and Space

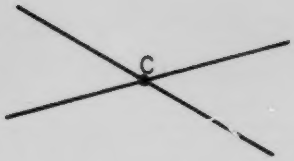
Next, we shall consider the idea of a line. In our study the term "line" shall mean "straight line." For us, a line will be a special type of set of points in space. Familiar examples of lines are suggested by:

1. the edge of a ruler
2. the crease in a carefully pressed pair of trousers
3. a radio antenna on a car
4. a flagpole

A geometric line extends without ending in each of two directions. If you were to use a ruler to draw a mark representing a line you could never find a ruler long enough. The mark representing a geometric line would have to extend both to the right and to the left beyond the longest ruler you could find.

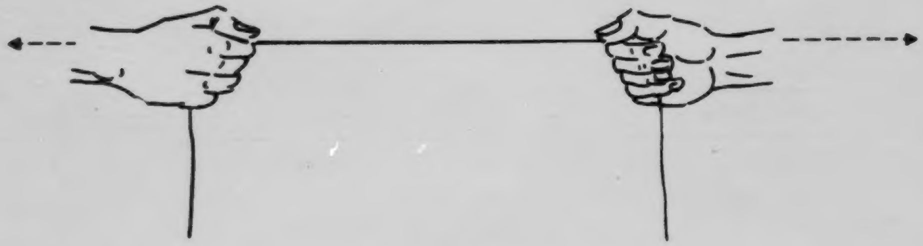
A line is suggested by the white stripe drawn down the middle of a perfectly straight highway. Considered as a geometric line the white stripe has no beginning and no end. It extends in both directions without limit.

Consider the line suggested by the top of this page. As a geometric line it extends without limit to the right and to the left of this book.



In the diagram at the right you see two lines which have the point C in common. If two different lines have one point in common, we say that the lines intersect. We call the point C the point of intersection.

Think about two boys holding a string stretched between them. When the string is stretched tight between the boys' hands, the stretched string represents a straight line. The line itself goes beyond where the boys hold the string. It extends without limit in each direction.



The boys' fingers suggest points. When the fingers (points) are in a certain position, is there more than one possible position for the stretched string? You probably think, "Of course not." And you are right. You now have discovered a basic property of space.

Property 1: Through any two different points in space there is exactly one line.

Represent two points on the chalkboard or on a sheet of paper. Can you draw more than one line through these points? Remember that "line" means "straight line."

Since we shall be talking about lines it is convenient to have a system of naming lines. Since a line is a special set of points we name a line by naming any two points on the line. The order in which the points are named does not matter.

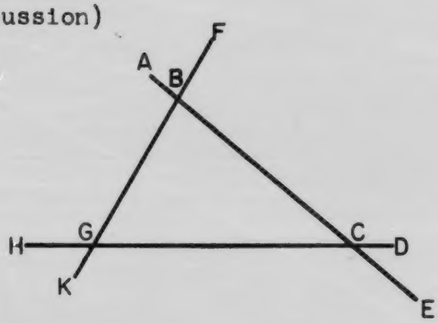
The line above is called "line RS" or "line SR." A symbol for this same line is $\overleftrightarrow{RS}$ or $\overleftrightarrow{SR}$. (Note: $\overleftrightarrow{RS}$ is read as "line RS".)

Exercises 4-1b

(Class Discussion)

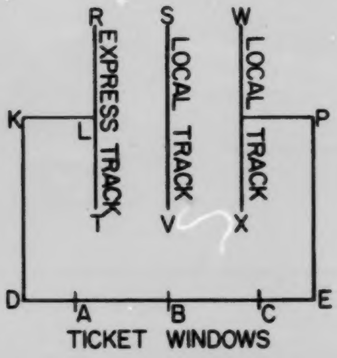
1. How many lines are drawn in the figure at the right? Name the lines.

2. In the same figure, name two lines that intersect. Name their point of intersection.



3. For party decorations, several crepe-paper ribbons were draped between two points on the gymnasium walls. Does this show, contrary to Property 1, that there may be several geometric lines through two points? Explain.

4. The drawing at the right represents a railroad station.
   a. Name a point that represents a ticket window.
   b. Name a line that represents a local track.
   c. Name the line on which the ticket windows are located.
   d. Name the line that represents the express track.
   e. Name two lines that intersect. Name their point of intersection.



We think of space as being a set of points. There are an unlimited number of points in space. Consider your living room at home. Suppose a mosquito is flying inside this room. Its location at a particular instant might be described as a point. All such points make up a set. All the points inside the room make up a portion of space. All the points outside the room make up another portion of space.

Exercises 4-1c
(Class Discussion)

1. Select a point on the floor of your classroom near the teacher's desk. Select another point on the rear wall of the classroom. Consider the line that contains these points.
   a. What objects in the room have a point in common with the line?
   b. What objects outside the room are also cut by the line?

2. A mathematical poet might say "Space is like the bristly, spiny, porcupine." In what ways is this a description of space?

3. Consider your school gymnasium.
   a. Name some objects that may represent points in the portion of space occupied by the gymnasium.
   b. Name some objects that may represent lines in the portion of space occupied by the gymnasium.

4. When a surveyor marks the boundaries of a piece of land, he places small stone blocks at the corners. A small hole or nail in the top of each block represents a point. If the stone blocks are not moved out of position, the original boundaries may be found at any later time. Explain why we can be sure of this.

4-2. Planes.

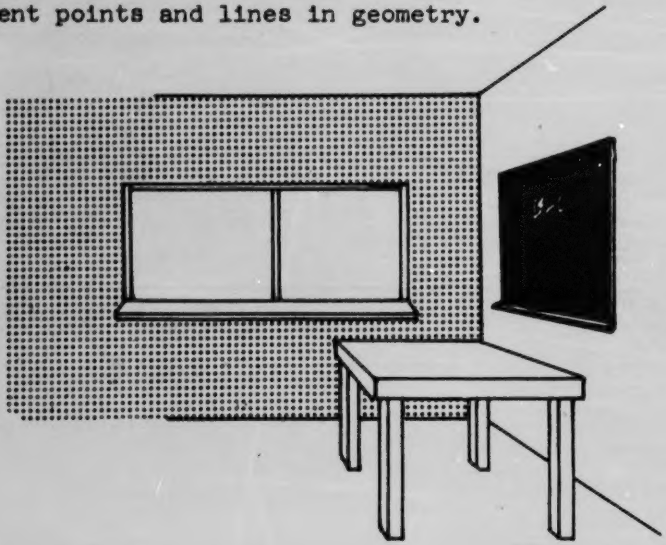
A plane in geometry is suggested by any flat surface. Familiar examples of planes are

1. the wall of a room
2. the top of a desk
3. a seesaw board
4. an auditorium platform

Choose some point of the plane suggested by the floor of your room. Think of lines through this point that also lie in the plane of the floor. In your imagination, follow these lines in all possible directions. The paths of these lines will remain in the plane but they will have no endings. A plane, then, has no boundaries.

We think of a plane as containing many points and many lines. As you look at the wall you can think of many points on it. You can also think of the many lines containing these points.

The points where the side wall of a room joins the ceiling suggest a line in either the plane represented by the ceiling or the plane represented by the side wall. The edges of the chalk trough represent lines in the classroom. At least one of these is in the plane represented by the chalkboard. Any number of points and lines could be marked on the chalkboard to represent points and lines in geometry.



Mathematicians think of a plane as a set of points in space on a flat surface. It is not just any set of points but a special kind of set. We have already seen that a line is a special set of points in space. The special set of points making up a line is different from the special set of points making up a plane.

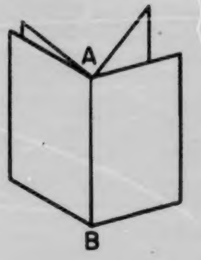
Exercises 4-2a

1. Describe some planes suggested by figures in your kitchen at home.
2. Consider the planes suggested by the walls and floor of your school library. In each of these planes, describe some lines and some points.
3. Why is the word "plane" used in the following names?--airplane, aquaplane, seaplane, carpenter's plane, plane sailing.

Think of two points marked on a chalkboard. The line through these points can also be drawn on the chalkboard. Can you connect these two points with a straight line that does not lie in the plane of the chalkboard? Do you agree that the line through these points must lie on the chalkboard?

Property 2: If a line contains two different points of a plane, it lies in the plane.

The figure at the right represents an open notebook. The points A and B represent the endpoints of the binding. The notebook with its pages spread apart suggests that there are many planes through the pair of points A and B.

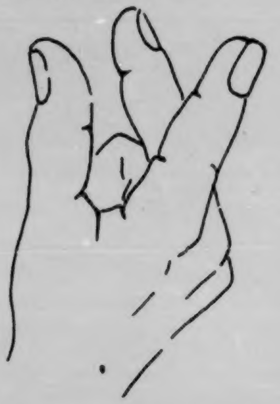


Consider the hinges of a door as a pair of points. The door itself suggests a plane. The door in various positions suggests many planes. Thus, we have many planes containing the two hinges (points).

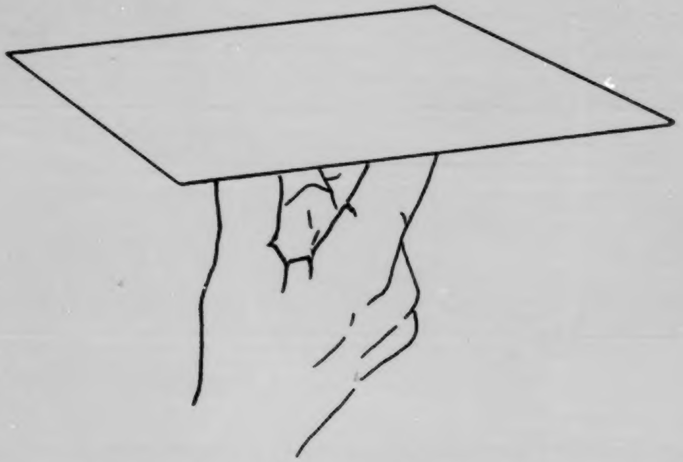
These two examples suggest **the following** "If we have two points, then many planes contain this pair of points."

Suppose we have three points not all on the same line. The bottoms of the legs of a tripod are an example of this. The bottoms of the legs of a three-legged stool are another example. Such a stool will stand firmly against the floor. A four-legged stool does not always stand firmly on all four legs unless it is carefully constructed.

Spread out the thumb and the first two fingers of your right hand as in the figure below. Hold them stiffly and think of their tips as being points.



Now, take a piece of cardboard and place it so that it lies on the tips of your thumb and two fingers (that is, on the three points).



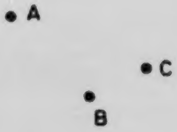
By pressing on the cardboard with your left hand, can you hold the cardboard against the tips of your thumb and two fingers? If you bend your wrist and change the position of your thumb and fingers, will the cardboard be in a new position? With your right hand in any one position, is there more than one way in which a flat surface can be held against the tips of your fingers? In each such case, the position of the cardboard is fixed by your thumb and two fingers. Do you agree that for each position of your right hand there is only one flat surface?

Property 3: Through any three points, not all on the same line, there is exactly one plane.

Can you use this property to explain the following? If the legs of a chair are not exactly the same length you are able to rest the chair on only three legs, but not on four.

Exercises 4-2b

(Class Discussion)



1. In a page of your notebook mark three points as shown. Label them A, B, and C. A, B, and C are in one plane.
2. Draw a line through points A and B. Draw a line through points B and C. Are $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ in the plane of A, B, and C? Why?
3. Mark a point R on $\overleftrightarrow{AB}$, and a point S on $\overleftrightarrow{BC}$.
4. Draw a dotted line through points R and S. R is in the plane of A, B, and C. Therefore, $\overleftrightarrow{RS}$ is in the plane of A, B, and C. Which property says this?
5. Take another pair of points K and L so that K is a point anywhere on $\overleftrightarrow{AB}$ and L is anywhere on $\overleftrightarrow{BC}$. Join this pair of points with dotted lines. Can you explain why line KL is in the plane of A, B, and C?
6. Repeat the above process with other pairs of points and lines.

The sets of points represented by the dotted lines are contained in the plane of A, B, and C. The plane which contains A, B, and C can now be described. It is the set of all points which are on the lines described as follows.

Each line contains two points of the figure. One of these points is on the line joining A and B. The other of these points is on the line joining B and C.

Exercises 4-2c

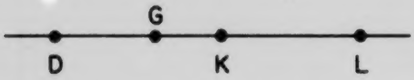
1. Class discussion exercise. A plane contains three points suggested by two front feet of the teacher's desk and the pencil sharpener.
   a. Through what objects in the room does this plane pass?
   b. Through what objects outside the room does this plane pass?
2. Class discussion exercise. Consider the following points suggested by:
   X-the foot of a chair in your living room.
   Y-the electric light switch in your living room.
   Z-the top of your TV aerial.
   W-the knob on the door to your bedroom.
   How many planes are suggested by these points? Explain your answer.
3. Photographers, surveyors, and artists often use tripods to support their equipment. Why is a three-legged stand a better choice for this purpose than one with four legs?
4. Point A is suggested by the top of a flagpole.
   Point B is suggested by the tip of your left shoe.
   Point C is suggested by the tip of a church steeple.
   a. How many different planes may contain A and B?
   b. How many different lines may contain A and C?
   c. How many different planes may contain A, B, and C?
5. Consider the plane suggested by a sidewalk of a city street. Also, consider the following points:
   the foot of a pole holding the street sign (A)
   the right foot of a person standing still and looking at the store window (B).
   the top of a telephone pole (C)
   the point where the rear wheel of a parked bicycle touches the sidewalk (D)
   Which lines passing through two of these points lie in the plane of the sidewalk of the street? Explain.
6. BRAINBUSTER. How many lines can be drawn through four points, a pair of them at a time, if the points:
   a. lie in the same plane, with no three in a straight line?
   b. do not lie in the same plane?

4-3. Names and Symbols.

We have seen that a point is represented by a dot and named by using a capital letter. It is important to understand that the dot merely represents a point but is not the point itself. The dot mark that represents a point has size on the paper. The point itself has no size.

In the same spirit, we represent a line in the following ways or . Recall that "line" for us means "straight line." Also, recall that a line extends without limit in two directions. Here again, the line drawn on the paper has width. Geometric lines, however, have no actual width. The mark on the paper merely represents a line.

A line is a special set of points. We name a line by picking any two of the points on the line. The line below



may be named in any of the following ways:

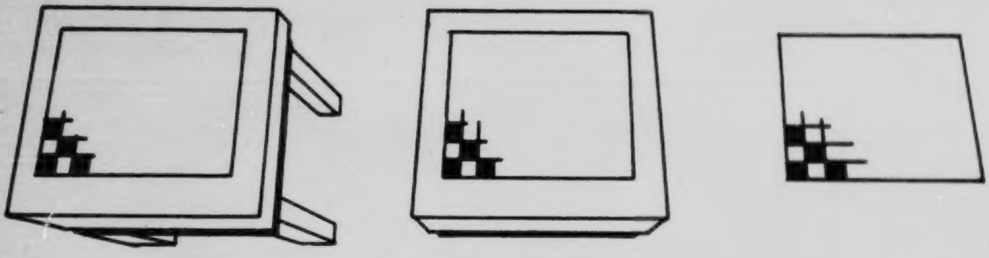
"line DG" or $\overleftrightarrow{DG}$ or $\overleftrightarrow{GD}$

"line LD" or $\overleftrightarrow{LD}$ or $\overleftrightarrow{DL}$

"line GK" or $\overleftrightarrow{GK}$ or $\overleftrightarrow{KG}$

Can you name this line in other ways?

Notice how frequently the word "represent" appears in these explanations. A point is merely represented by a dot because as long as the dot mark can be seen, it has size. But a point, in geometry, has no size. Also lines drawn with chalk are rather wide, wavy, and not really straight. Are actual geometric lines like this? Recall that "line" for us means "straight line." A drawing of a line by a very sharp pencil on very smooth paper is more like the idea of a line. Yet, if you look at it through a magnifying glass, you will see that it is far from perfect. Thus, by a dot we merely show the position of a point. A drawing of a line merely represents the line. The drawing is not the actual line.

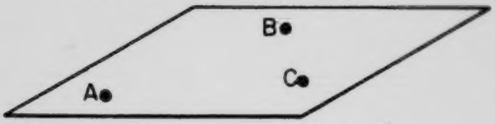


(a) (b) (c)

Just as we need to represent point and line, we find it necessary to represent a plane. Figure (a) is a picture of a checkerboard resting on top of a card table. Figure (b) is the same with the legs removed. Figure (c) is the checkerboard alone.

The table top suggests a portion of a plane. In this case, the checkerboard suggests a smaller portion of a plane and the sections in figure (c) represent still smaller portions of the plane.

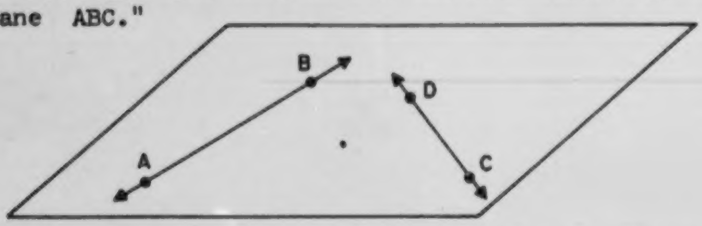
We may want to make a drawing of a plane which is not the same as the plane of the paper. It is customary to use a diamond-shaped figure as shown below. The plane itself extends beyond the lines shown in the figure.



In the figure above, do points A and C seem closer to you than point B? If not, imagine point B as an opponent's checker at the far edge of the checkerboard. Then A and C would be checkers belonging to you.

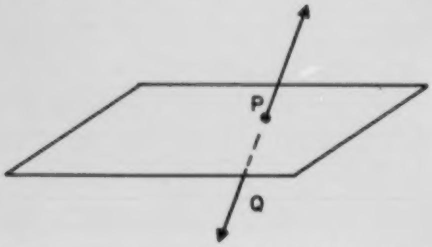
Points of a plane are indicated by dots and named by capital letters as before. Recall that Property 3 states, "Any three points not all on the same line are in exactly one plane."

Therefore, a plane is usually named by naming three points in the plane. For example, the plane on the preceding page is called "plane ABC."



In the above figure, A, B, C, and D are considered to be points in a plane. A line is drawn through A and B. Another line is drawn through C and D. According to Property 2, what can be said about $\overleftrightarrow{AB}$? What can be said about $\overleftrightarrow{DC}$?

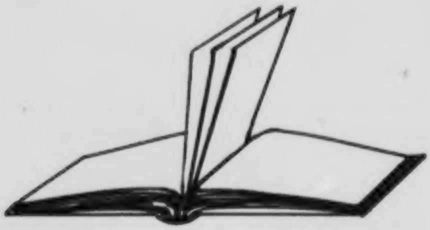
It is possible that a line might pierce or puncture a plane. A picture of this situation may appear thus:



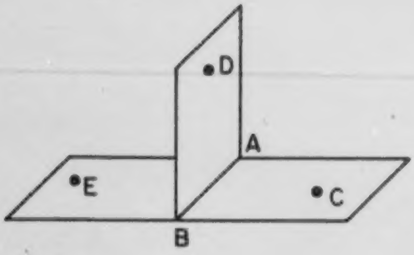
Suppose the plane were a solid table top. Then the dotted portion of $\overleftrightarrow{PQ}$ would be hidden from view.

Once again, we see that the drawing only represents the situation.

This is a drawing of an open book.



If all pages of the book except one were removed, then the drawing would look like this.



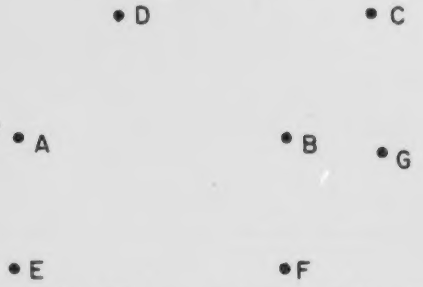
This figure suggests the meeting, or intersection, of two planes. The plane that appears to lie flat on a table contains the front and back covers. The plane that appears to stand up contains the single page. Let us see how these intersecting planes may be named.

The plane that appears upright may simply be called "plane ABD." The plane that appears flat may be named in several ways. We may call it "plane ABE" or "plane ABC." Can you name this plane in other ways?

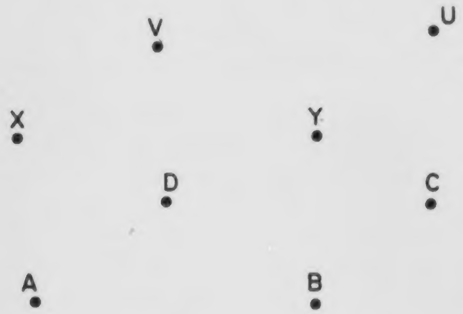
Remember that this plane of the book cover is a set of points extending beyond the cover. Since plane ABC and plane ABE are different names for this set of points, we write plane ABC = plane ABE. Here the equal sign means that both sets contain the same elements.

Exercises 4-3

1. Transfer the points in the figure below to a piece of paper by tracing. From now on we shall refer to the copy you have made. Join A to B, then B to C, C to D, D to A in that order. Now join A to E, B to F, C to G. What familiar piece of furniture might this sketch represent?
   a. Name the plane suggested by the top of the object.
   b. Name a line suggested by an edge of the object.
   c. Name a plane that contains this line.
   d. If line CF is drawn in which plane will it be contained?

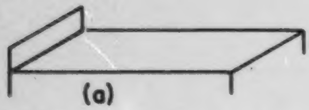


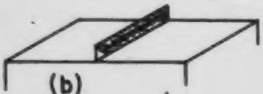
2. Make a tracing of the figure below. Join A to B, B to C, C to D, D to A, A to X, B to Y, D to V, and C to U. What has happened to the table?

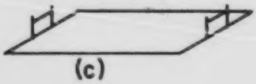


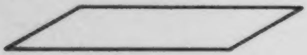
3. Each of the sketches below represents one of the familiar objects listed below. Match the sketches with the names.

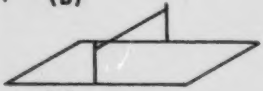
Cot	Football field	Line of laundry
Ping pong table	High jump	Ladder
Carpet	Coffee table	Open door
Chair		Shelf

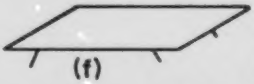

(a)

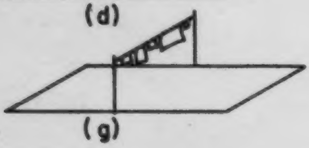

(b)

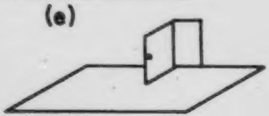

(c)

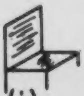

(d)

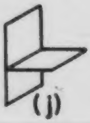

(e)

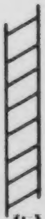

(f)


(g)


(h)

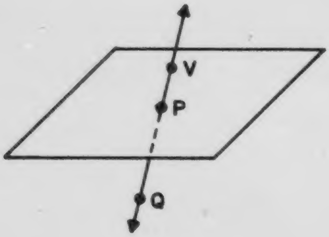

(i)


(j)


(k)

4. Try to draw
   a. a lump of sugar.
   b. a lunch box.
   c. a desk.
   d. a tent.
   e. an Egyptian pyramid. (Its base is a square.)

5. In the figure below, is point V a point of $\overleftrightarrow{PQ}$? Is point Q an element of the plane? Is V? How many points of $\overleftrightarrow{PQ}$ are elements of the plane?



6. **Figure (b) is a copy of figure (a) except for labeling.**

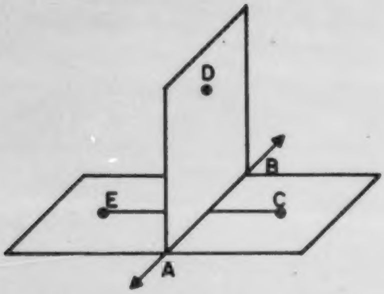


figure (a)

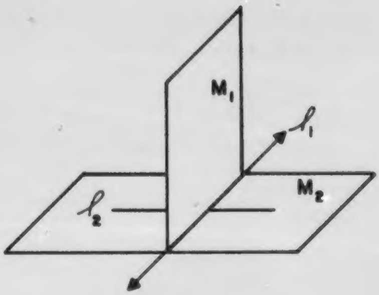


figure (b)

Two boys named "Tom," who are in the same class might be called T_1 and T_2, to avoid confusing one with the other. Similarly, two different lines may be denoted as ℓ_1 and ℓ_2. The small numbers are not exponents. They are called subscripts. Plane ABD in figure (a) corresponds to M_1 in figure (b). $\overleftrightarrow{AB}$ in figure (a) corresponds to ℓ_1 in figure (b).

In the left-hand column are listed parts of figure (a). Match these with parts of figure (b) listed in the right-hand column.

Parts of Figure (a)

1. $\overleftrightarrow{EC}$
2. Plane ABC
3. Plane ABD
4. Plane EBA
5. $\overleftrightarrow{AB}$
6. The intersection of plane ABC and plane ABD.

Parts of Figure (b)

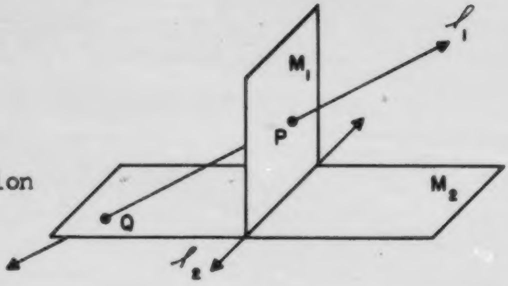
a. ℓ_1

b. ℓ_2

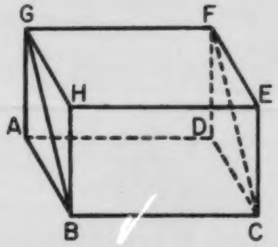
c. M_1

d. M_2

7. In the figure at the right:
   a. Does ℓ_1 pierce M_1?
   b. Does ℓ_1 pierce M_2?
   c. Is ℓ_1 the only line through P and Q?
   d. What is the intersection of M_1 and M_2?
   e. Is ℓ_1 in M_2?
   f. Would ℓ_1 meet ℓ_2? Are ℓ_1 and ℓ_2 in the same plane?



8. The figure at the right represents a wooden packing box.
   a. Name the plane represented by the bottom of the box.
   b. Name the plane represented by the right side of the box.
   c. Name the line represented by the brace on the left side.
   d. Name two intersecting planes.



4-4. Intersection of Sets.

In Chapter 3 we learned that the word "set" is used to talk about collections of objects. For example, the set of members of the crew in an airliner is composed of pilot, copilot, navigator, radio operator, and stewardess.

In written work, we usually use braces to indicate sets and refer to sets by capital letters. For example:

If S is the set of even numbers greater than 0 and less than 10, we write

$$S = \{2, 4, 6, 8\}.$$

If T is the set of outfielders on a baseball team, we write

$$T = \{\text{left fielder, center fielder, right fielder}\}.$$

A set may have no elements. The set with no elements is called the empty set. For example, the set of Congressmen less than fifteen years of age is the empty set since there are no Congressmen who are less than fifteen years of age.

Sometimes, we have two sets which have elements in common. For example: let set $A = \{\text{red, blue, green}\}$;

let set $B = \{\text{yellow, green, orange, blue}\}$.

The elements blue and green are in both sets A and B.

If we let set $C = \{\text{blue, green}\}$ then we call set C the intersection of sets A and B. Thus, the intersection of two sets is a set composed of the elements which are common to the two sets.

We use the symbol $\cap$ to mean the intersection of two sets. In the example above, we have

$C = A \cap B$ (Read: Set C is the intersection of sets A and B, or C is the intersection of A and B.)

If two sets have no elements in common, then the intersection is the empty set.

For example: let set X be the set of ball players on the Giants team;
let set Y be the set of ball players on the Dodgers team.

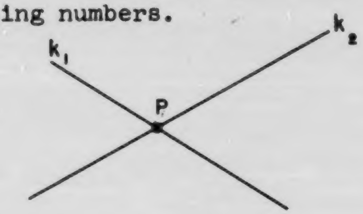
Then $X \cap Y$, the intersection of X and Y, is the empty set.

Exercises 4-4a

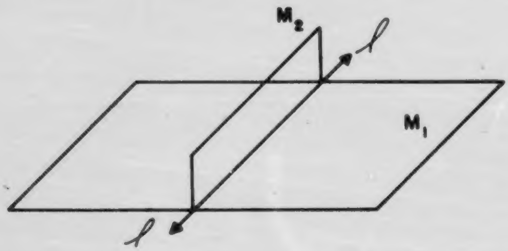
1. Write the elements of the set whose members are:
    a. The days of the week.
    b. The subjects you are studying this term.
    c. The members of your family.
    d. The students in your class who are more than 8 feet tall.
    e. The whole numbers greater than 17 and less than 23.
2. Write three elements of each of the following sets:
    a. States in the United States.
    b. Months of the year.
    c. Whole numbers divisible by 5.

3. In each case, write the elements of the set that is the intersection of the given sets.
   a. A = {1, 3, 5, 7, 9, 11}
      B = {1, 4, 9, 16}
   b. P = {John, Ethel, Bill, Frank, Alice}
      Q = {Frank, Paul, Alice, Diane, John, Helen}
   c. X = The whole numbers 2 through 12.
      Y = The whole numbers 9 through 20.
   d. The letters used in your full name and the letters used in the name of your school.
   e. The set of national holidays and the set of days in July.
   f. The set of countries in North America and the set of countries in South America.

4. Give the elements of the intersection of the following pairs of sets.
   a. The whole numbers 2 through 32 and the whole numbers 9 through 20.
   b. The members of your class and the girls with blonde hair.
   c. The whole numbers and the counting numbers.
   d. The set of points on line k_1 and the set of points on line k_2.



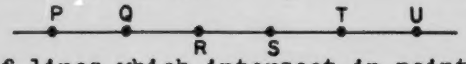
Sets are useful in talking about figures in geometry. In the figure below, M_1 represents a ping-pong table and M_2 represents the net.



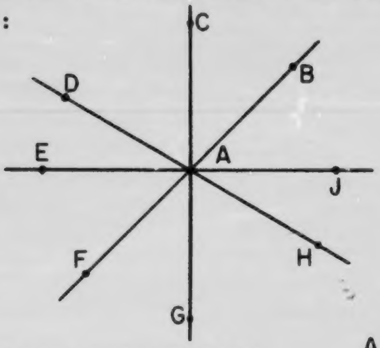
This figure suggests two planes M_1 and M_2. The line called ℓ seems to be in M_1 and also in M_2. Every point in M_1 which is also in M_2 seems to be on the line ℓ. Thus, the intersection of planes M_1 and M_2 seems to be ℓ. This may be written $M_1 \cap M_2 = \ell$.

Exercises 4-4b

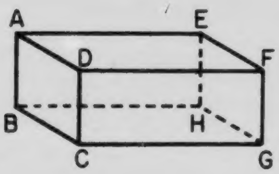
1. Write three elements of each of the following sets.
   a. The set of points on the line below, some of which are labeled in the figure.



   b. The set of lines which intersect in point A in the figure below:



   c. The set of planes suggested by the figure at the right. (a chalk box)

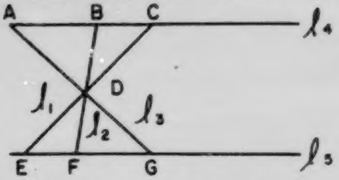


   d. Choose two planes in the figure. What is the intersection of the planes you have chosen? Do this again for three other pairs of planes.

   e. Choose two lines in the figure. What is the intersection of the lines you have chosen? Do this again for three other pairs of lines.

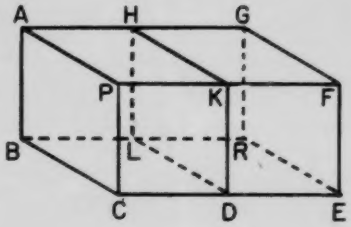
2. Write the intersections of the given pairs of sets:

   a. ℓ_3 and ℓ_5.
   b. ℓ_1 and ℓ_3.
   c. ℓ_4 and ℓ_3.
   d. ℓ_4 and ℓ_5.



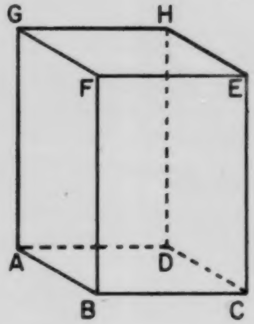
3. Describe or list the elements of the intersections of the following pairs of sets. This is a sketch of an orange crate.

   a. The set of points on plane HKL and the set of points on plane APF.
   b. The set of points on plane HKL and the set of points on line BR.
   c. The set of points on plane BRC and the set of points on plane AGF.
   d. The set of points on line PC and the set of points on plane APF.



4-5. Intersections of Lines and Planes.

Two Lines



On the right we have a drawing of a tool box. Think of the edges ($\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{EC}$, $\overleftrightarrow{GH}$, etc.) as representing lines. Some of these lines intersect and some do not.

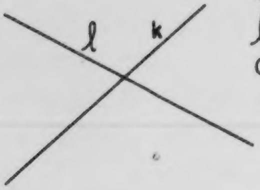
Let us look at the plane GHE suggested by the top of the box and the plane ABD suggested by the bottom of the box. In these planes we find:

1. Pairs of intersecting lines ($\overleftrightarrow{AD}$ and $\overleftrightarrow{CD}$, $\overleftrightarrow{GF}$ and $\overleftrightarrow{FE}$). What is their intersection? Can you find other pairs?

2. Pairs of lines which do not intersect but have the same direction ($\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$, $\overleftrightarrow{FE}$ and $\overleftrightarrow{GH}$). Can you find other pairs?

Can you hold your two arms in a position so that they represent intersecting lines? Can you hold them in a position so that they do not intersect but have the same direction? Can you hold them in a position so that they do not intersect and do not have the same direction? Can you hold your two arms so that they represent straight lines in any other kind of position?

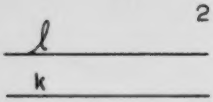
The possible positions of 2 different lines may be divided into 3 cases:



1. ℓ and k intersect, ($\ell \cap k$ is not the empty set.). ℓ and k cannot contain the same two points. Can you explain why?

   (Hint: Suppose ℓ and k did contain the same two points, what would follow by Property 1?)

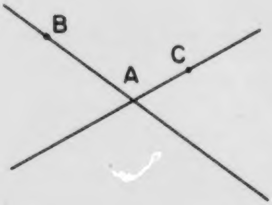
   The intersection of ℓ and k is a point.



2. ℓ and k do not intersect and are in the same plane. $\ell \cap k$ is the empty set. ℓ and k are said to be parallel.

3. ℓ and k do not intersect and are not in the same plane. It is difficult to draw this on paper. $\ell \cap k$ is the empty set and ℓ and k are not in the same plane. ℓ and k are said to be skew lines.

In this figure are shown two lines which intersect at point A. B is a point on one of the lines. C is a point on the other line. There is exactly one plane which contains A, B, and C. Which property tells us this?



Which property tells us that $\overleftrightarrow{AB}$ is in this plane? Is $\overleftrightarrow{AC}$ in this plane? Why? Therefore, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ lie in this plane. There is only one plane which contains both $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$. We thus have

Property 3a: *If two different lines intersect, exactly one plane contains both lines.*

Oral Exercises 4-5a

1. Using two pencils to represent lines, illustrate the following:
   - a. Two parallel lines
   - b. Two skew lines
   - c. Two intersecting lines
2. Figure 1 represents a deck chair. In this drawing:
   - a. Name two intersecting lines. What is their intersection?
   - b. Name two parallel lines.
   - c. Name two skew lines.

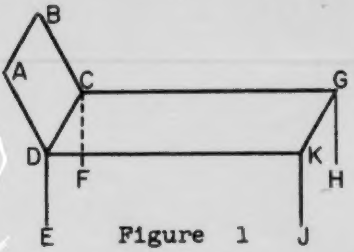


Figure 1

3. In Figure 1, use Property 3a to explain why $\overleftrightarrow{CD}$ and $\overleftrightarrow{CG}$ are in the same plane.
4. In Figure 1, Use Property 3 to explain why points A, D, and C are in the same plane.
5. Figure 2 represents a folding chair. In this figure
   - a. Name four planes.
   - b. Name three pairs of skew lines.
   - c. If the chair stands firmly on the ground, what can be said about points G, H, J, and K?

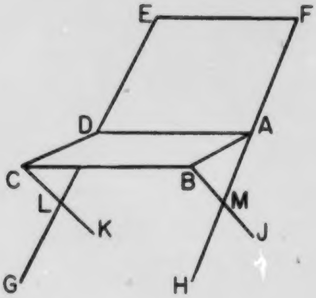
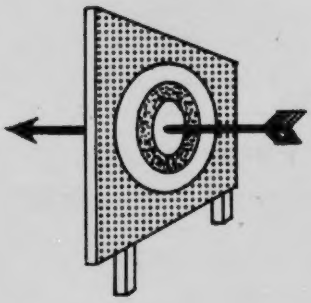
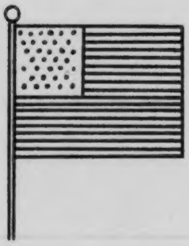


Figure 2

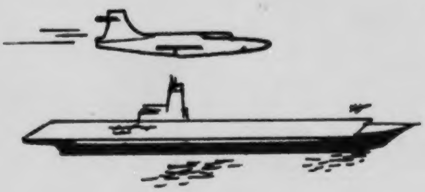
A Line and A Plane



A line and a plane may be situated so that their intersection is only one point. This is shown in the figure at the right.



There is another way a line and a plane may be situated so that their intersection contains many points. This is shown in the figure at the right. In this case, every point of the line is in the plane. We say that the line lies in the plane.



There is still another way a line and a plane may be situated so that their intersection contains no points at all. This is shown in the figure at the right.

Thus, a line and a plane may:

1. intersect in a point;
2. intersect in many points;
3. have no point of intersection.

Two Planes

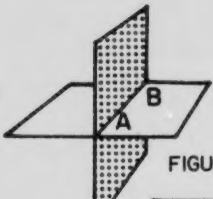


FIGURE 1

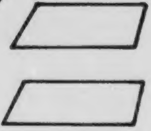


FIGURE 2

Let us think of two different planes in space. Such planes may intersect as in Figure 1, or the intersection of these two planes may be the empty set, as in Figure 2.

Consider the plane of the front wall and the plane of a side wall in the same room. You will notice that they intersect in more than one point.

Take two sheets of paper and hold one sheet in each hand. The sheets of paper can be held so that they have only one point of intersection. Now, let us consider the *planes* of the sheets of paper and not just the sheets themselves. Remember these planes extend without limit in all directions. We see that, if the planes have one point in common, then their intersection will contain other points.

Keep the sheets of paper flat. Can you hold them so that the plane they suggest will intersect in only two points?

Keep the sheets flat. Can you hold them so that they intersect in a curved line?

In Figure 1, let A and B be two points, each of which lies in the two intersecting planes. Property 2 states that if a line contains two different points of a plane, then it lies in the plane. According to this property $\overleftrightarrow{AB}$ must lie in each of the planes. Hence, the intersection of the two planes is a line.

But, if the intersection contains any other point not on $\overleftrightarrow{AB}$, then the two intersecting planes must be the same plane. Property 3 states that any three points not on the same line are in only one plane. We now state

Property 4: *If the intersection of two different planes is not empty, then the intersection is a line.*

If the intersection of two planes is the empty set, then the planes are said to be parallel. Examples of pairs of parallel planes are suggested by shelves in a bookcase. Can you find examples of parallel planes

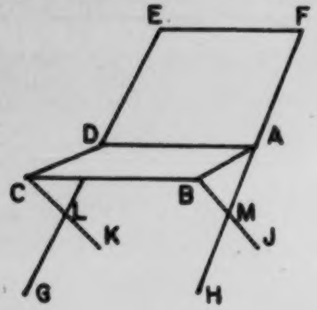
a. in your classroom?
b. at home?

Thus, in discussing the intersection of two planes we have the following cases. If M and N represent two planes:

1. $M \cap N$ is not empty (the planes intersect). $M \cap N$ is a line.
2. $M \cap N$ is empty (the planes are parallel). Are there any other cases? Why?

Oral Exercises 4-5b

The figure below represents a folding chair. By using our new notation, we can ask questions in a very simple way about the lines, planes, and points in the figure. First, read each question aloud and then choose the best response from the list on the right.

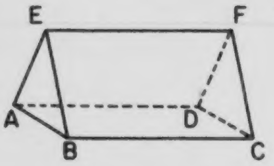


1. What is $\overleftrightarrow{FH} \cap \overleftrightarrow{BJ}$?
2. What is $\overleftrightarrow{AM} \cap \overleftrightarrow{JM}$?
3. Name $\overleftrightarrow{CD} \cap \overleftrightarrow{AB}$.
4. Name $\overleftrightarrow{AB} \cap \overleftrightarrow{EG}$.
5. Plane ABC $\cap$ $\overleftrightarrow{EG}$ = ?
6. Plane ABC $\cap$ $\overleftrightarrow{EF}$ = ?
7. Plane ABC $\cap$ $\overleftrightarrow{AD}$ = ?
8. Plane DEF $\cap$ plane ABC = ?
9. Plane ABC $\cap$ plane HMJ = ?
10. Plane GLK $\cap$ plane HMJ = ?
11. Plane BCD is the same as ?
12. Plane ACD = ?

a. The empty set
b. A
c. M
d. D
e. L
f. $\overleftrightarrow{AD}$
g. $\overleftrightarrow{AB}$
h. Plane MAB
i. Plane ABC
j. None of these

Exercises 4-5c

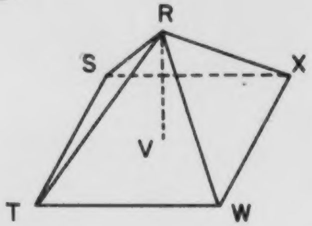
1. In the tent figure at the right
   a. Name three planes
   b. Name two skew lines.
   c. Name two parallel lines.



2. In the same figure:
   a. Name two parallel planes.
   b. Name two intersecting planes.
   What is their intersection?
   c. Name two intersecting lines.
   What is their intersection?
   d. Name the intersection of planes AEB, ABC, and AED.

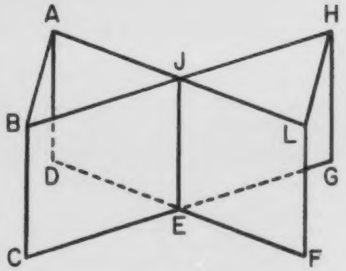
3. In the same figure:
   a. Name $\overleftrightarrow{AE} \cap \overleftrightarrow{EB}$.
   b. Name $\overleftrightarrow{EF} \cap$ plane FDC.
   c. Name $\overleftrightarrow{EF} \cap \overleftrightarrow{BC}$.
   d. Name plane ABC $\cap$ plane EFC.
   e. Name plane AEB $\cap$ plane ADC.
   f. Name plane AEB $\cap$ plane DFC.
   g. Name $\overleftrightarrow{EF} \cap$ plane ABC.

4. The figure at the right is a tent with $\overleftrightarrow{RV}$ the center pole. In this figure name the following:
   a. $\overleftrightarrow{RV} \cap$ plane STW
   b. $\overleftrightarrow{RW} \cap \overleftrightarrow{WX}$
   c. $\overleftrightarrow{SX} \cap \overleftrightarrow{RV}$
   d. $\overleftrightarrow{RV} \cap \overleftrightarrow{TW}$
   e. Plane RST $\cap$ plane WTR
   f. Plane SXT $\cap$ plane RTW

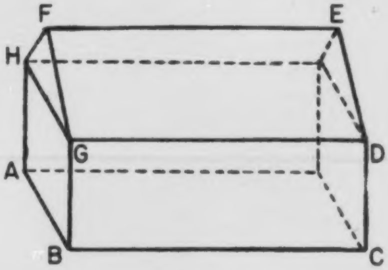


5. Property 3a states that, "If two different lines intersect, exactly one plane contains both lines." Using the figure for Problem 4 name three pairs of intersecting lines and the planes that contain each pair of lines.

6. In the figure at the right:
   a. Name plane BCE $\cap$ plane ADF.
   b. Name two skew lines.
   c. Name $\overleftrightarrow{BH} \cap \overleftrightarrow{CG}$.
   d. Name $\overleftrightarrow{BH} \cap \overleftrightarrow{AL}$.
   e. Name the plane containing $\overleftrightarrow{BH}$ and $\overleftrightarrow{AL}$.
   f. Name $\overleftrightarrow{CE} \cap$ plane ADF.
   g. Name the intersection of plane CJG and $\overleftrightarrow{DF}$.



7. Consider this sketch of a barn. We have labeled eight points on the figure. Think of the lines and planes suggested by the figure. Name the following:
   a. A pair of parallel planes.
   b. A pair of planes whose intersection is a line.
   c. Three planes that intersect in a point.
   d. Three planes that intersect in a line.
   e. A line and a plane whose intersection is empty.
   f. A pair of parallel lines.
   g. A pair of skew lines.
   h. Three lines that intersect in a point.
   i. Four planes that have exactly one point in common.



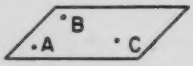
4-6. Summary.

The part of mathematics that deals with space and location in space is called geometry. In this chapter, we considered non-metric or "no measurement" geometry. The ideas of point, line, plane, and space are some of the key ideas used in non-metric geometry.

The ideas of a point, line and plane are suggested by many objects around us. For example, the idea of a point is suggested by a doorknob, the tip of a man's collar, and the pitcher's mound in baseball. The idea of a line is suggested by a flagpole, a telephone line, and a high jump bar. The idea of a plane is suggested by a flat surface, such as a ramp, the side wall of a living room, and the top of a desk. We think of a line in geometry as extending in both directions without limit. Like a line in mathematics, a plane is thought of as being unlimited in extent. A plane has no boundaries.

Marks on paper or the chalkboard are used to represent a point, a line, and a plane. A point is represented by a dot and named with a capital letter, as . A (point A). A line is represented by

or by

. This is read "line CD" ($\overleftrightarrow{CD}$). A plane is represented by 

and is read "plane ABC." It is important to remember that these drawings merely represent a point, a line, and a plane.

The following are important properties of points, lines, and planes.

Property 1: Through any two different points in space there is exactly one line.

Property 2: If a line contains two different points of a plane, it lies in the plane.

Property 3: Through any three points, not all on the same line, there is exactly one plane.

The intersection of two sets is a set composed of the elements which are common to both. We use the symbol $\cap$ to mean the intersection of two sets. The idea of intersection is important in dealing with geometric figures.

Two Lines

The possible positions of two different lines may be divided into three cases. If ℓ and k are lines, then

1. ℓ and k may intersect. $\ell \cap k$ is a point.
2. ℓ and k do not intersect and are in the same plane. $\ell \cap k$ is the empty set. ℓ and k are said to be parallel.
3. ℓ and k do not intersect and are not in the same plane. $\ell \cap k$ is the empty set. ℓ and k are said to be skew lines.

A Line and a Plane

A line and a plane may have three different positions. If ℓ is a line, and ABC is a plane, then they may:

1. intersect in a point. $\ell \cap$ plane ABC is a point.
2. intersect in many points. $\ell \cap$ plane ABC is many points. The line lies in the plane.
3. have no point of intersection. $\ell \cap$ plane ABC is the empty set.

Two Planes

There are two possible positions for two different planes: If ABC and RST are planes, then they may:

1. intersect in a line. Plane ABC $\cap$ plane RST is a line.
2. have no intersection. Plane ABC $\cap$ plane RST is the empty set. Plane ABC and plane RST are parallel.

The following are important properties of intersections.

Property 3a: If two different lines intersect, exactly one plane contains both lines.

Property 4: If the intersection of two different planes is not empty, then the intersection is a line.

Some of the important terms introduced in this chapter follow.

EMPTY SET: The empty set is the set with no elements.

INTERSECTION OF SETS: The intersection of two sets is the set composed of the elements which are common to the two sets. The symbol used for the intersection of sets is $\cap$.

INTERSECTIONS:
1. Two different lines. The intersection of two different lines may be no points, or one point.
2. A line and a plane. The intersection of a line and a plane may be no points, one point, or many points. (The line lies in the plane.)
3. Two different planes. The intersection of two planes is no points or a line.

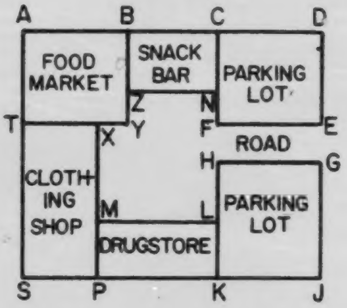
PARALLEL LINES: If two different lines are in the same plane and do not intersect, they are said to be parallel.

SKEW LINES: If two different lines are not in the same plane and do not intersect, they are said to be skew lines.

4-7. Chapter Review.

Exercises 4-7

1. At the right is a diagram of a shopping center. In this diagram:
   - a. Name the line that represents the front of the snack bar.
   - b. Name a point on the entrance road.
   - c. What is $\overleftrightarrow{LK} \cap \overleftrightarrow{MP}$?
   - d. What is $\overleftrightarrow{PM} \cap \overleftrightarrow{TY}$?
   - e. Name a line that represents a side of the entrance road.
   - f. What is $\overleftrightarrow{NF} \cap \overleftrightarrow{AD}$?
   - g. What is $\overleftrightarrow{HG} \cap \overleftrightarrow{SP}$?

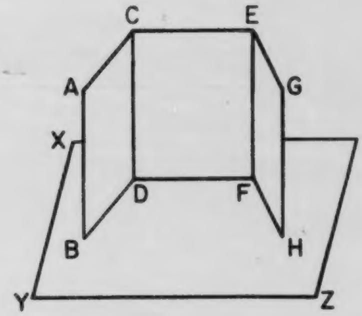


SHOPPING CENTER

2. Consider the following points in a park suggested by:
   a. A foot of a park bench (A).
   b. The top of the tallest tree in the park (B).
   c. The top of the flagpole (C).
   d. A headlight of a parked car (D).

   How many planes are suggested by these points? Explain your answer.

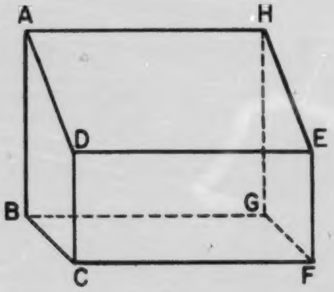
3. The figure at the right is a sketch of a baseball backstop.
   a. Name three planes in the figure.
   b. Name two planes that intersect and name their intersection.
   c. Name two skew lines.
   d. What is $\overleftrightarrow{BD} \cap \overleftrightarrow{DF}$?
   e. What is $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$?
   f. What is $\overleftrightarrow{FH} \cap$ plane ECD?
   g. What is $\overleftrightarrow{CD} \cap \overleftrightarrow{YZ}$?
   h. What is $\overleftrightarrow{EF} \cap$ plane ABD?



4. Consider the plane suggested by the playing field of a major league ball park. Also, consider the following points:
   a. The home plate (R).
   b. The microphone in the radio broadcaster's booth (S).
   c. The intersection of the right field foul line and the right field fence (T).
   d. Second base (V).

   Which lines passing through two of these points lie in the plane of the playing field of the ball park. Why?

5. The figure at the right is a sketch of a shed.



   a. Name the plane representing the top of the shed.
   b. Name a plane representing a side wall of the shed.
   c. What is $\overleftrightarrow{AD} \cap \overleftrightarrow{BG}$?
   d. What is plane BGF $\cap$ plane AHG?
   e. What is $\overleftrightarrow{GF} \cap$ plane DCF?
   f. What is $\overleftrightarrow{CD} \cap$ plane BCF?
   g. What is $\overleftrightarrow{AB} \cap \overleftrightarrow{CB}$?
   h. What is plane AHG $\cap$ plane DEF?
   i. What is $\overleftrightarrow{AD} \cap \overleftrightarrow{HE}$?
   j. How many lines can be drawn containing points D and H? Explain.
   k. If a line is drawn containing points D and H, in what plane will this line lie?
   l. Name two planes in which $\overleftrightarrow{CD}$ lies.

4-8. Cumulative Review.

Exercises 4-8

1. How many number symbols are needed to write numerals in a base eight system?

2. Write the following numerals in words.
   a. 2,042 b. 37,256

3. Write in expanded form, using exponents.
   a. 3407 b. 11111_{two} c. 2143_{five}

4. What is the value of each of these expressions?
   a. 3^4 b. 9^2 c. 4 to the third power

5. Is this statement true? "The numeral following 33_{four} is 100_{four}."

6. Which is larger 2^5 or 5^2? How much larger?

7. In which number base has this multiplication been performed?

$$\begin{array}{r} 123 \\ \underline{32} \\ 312 \\ \underline{1101} \\ 11322 \end{array}$$

8. The number of counting numbers between 3 and 5 is ________.

9. The value of 1^{30} is ________. Give a reason for your answer.

10. Give a word description of the following set.

$\{3, 6, 9, 12, 15 \ldots\}$

11. Are the following sets closed uhder addition?
   a. $\{2, 4, 6, 8, 10 \ldots\}$
   b. $\{7, 14, 21, 28, 35 \ldots\}$

12. Use the distributive property to rewrite each of the following.
   a. $9 \cdot (3 + 2)$
   b. $(7 + 11) \cdot 6$

13. Which of the following is a true statement?
   a. All counting numbers are whole numbers.
   b. All whole numbers are counting numbers.
   c. Zero has no meaning.

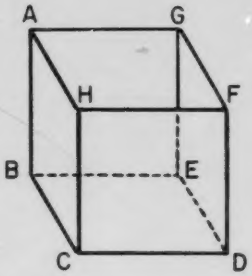
14. Write in symbols.
   a. 6 is less than 8.
   b. 9 is to the left of 12 on the number line.
   c. 3 is greater than 0.
   d. 15 lies between 13 and 17.

15. Use the associative property of multiplication so that the product can be found easily: $31 \times 5 \times 2$.

16. In each case, what are the elements of the set that is the intersection of the given sets?
    a. The set of even numbers less than 25 and the set of multiples of 3.
    b. The set of baseball players now in the National League and the set of baseball players now in the American League.

17. The following points are suggested by objects in your kitchen. Point A is suggested by the refrigerator handle. Point B is suggested by the foot of the kitchen table. Point C is suggested by the faucet on your kitchen sink.
    a. How many different planes contain A and B?
    b. How many different lines contain A and C?
    c. How many different planes contain A, B, and C?

18. The figure at the right represents an empty packing carton.
    a. How many planes are suggested by points A, H, and E?
    b. How many planes are suggested by points A, B, C, and E? Name these planes.
    c. What is plane AGH $\cap$ $\overleftrightarrow{CH}$?
    d. What is plane GFE $\cap$ plane BCD?
    e. What is plane AHF $\cap$ plane ECD?



Chapter 5

FACTORING AND PRIMES

5-1. The Building Blocks of Arithmetic.

We use counting numbers so often that we simply take them for granted. Where did they really come from?

We can think of the set of counting numbers as built up by starting with 1. By adding ones we get each following number in turn. This is the way we do it:

$$1 + 1 = 2 \qquad 1 + 1 + 1 + 1 = 4$$

$$1 + 1 + 1 = 3 \qquad 1 + 1 + 1 + 1 + 1 = 5$$

and so on, as far as you please.

Then 1 is sometimes called a building block because it is used to build all our counting numbers. Only 1 can be used to build all our counting numbers by addition.

Now suppose we try to build the counting numbers with multiplication by using 1. We get

$$1 \cdot 1 = ? \qquad 1 \cdot 1 \cdot 1 = ? \qquad 1 \cdot 1 \cdot 1 \cdot 1 = ?$$

We find that we always get 1 and never 2 or 3 or any other counting number. We cannot use multiplication by 1 to build our numbers.

If we use 2 as a multiplication building block, what numbers do we get?

$$2 \cdot 2 = 4 \qquad 2 \cdot 2 \cdot 2 = ? \qquad 2 \cdot 2 \cdot 2 \cdot 2 = ?$$

We see that there are many counting numbers which we do not get when we use 2 and multiplication. For example, we do not get 3, or 5, or 7. There are many counting numbers which cannot be built at all by multiplying smaller counting numbers in any way.

Exercises 5-1

Express each of the following counting numbers as the product of two smaller counting numbers. If it cannot be done for some of the numbers, write "no." Example: a. $21 = 3 \cdot 7$ b. 37 no

1. 12
2. 36
3. 31
4. 7
5. 8
6. 11
7. 35
8. 5
9. 39
10. 42
11. 56
12. 41
13. 82
14. 95
15. 73
16. 84
17. 87
18. 97

5-2. Multiples.

In the **Exercises** 5-1 above, you found two kinds of numbers. Some were products of two smaller counting numbers and some were not. In order to talk about these two kinds of numbers, we need names for them. One of the words used is multiple. In $24 = 8 \cdot 3$ we say that 24 is a multiple of 8 and also a multiple of 3.

Definition: A multiple of a number is the product of that number and any whole number.

Example: 6 is a multiple of 3; 10 is a multiple of 5.
Some multiples of 7 are 0, 7, 21, 42, 63.

Exercises 5-2

Name three multiples of each number below. See the example above.

1. 5
2. 8
3. 4
4. 7
5. 12
6. 9
7. 6
8. 11
9. 25
10. Write the numerals from 1 to 20. Circle "2" and cross out all the other multiples of 2. The work has been started for you on the next line.

1 ② 3 ~~4~~ 5 ~~6~~ 7 8 9 10 11 12 13 14 15 16 17 18 19 20

11. Write the numerals from 1 to 20. Circle "3" and cross out all the other multiples of 3.

12. List the set of multiples of 13 less than 100.

13. a. Underline the numbers below that are multiples of 7.
    b. Circle the numbers below that are multiples of 9.

    7 18 40 90 84 19 47 63
    127 252 25 35 36 273 48 105

14. Write all multiples of 8 greater than 50 and less than 100.

15. BRAINBUSTER. Below is shown a multiplication problem in which each digit has been replaced by a letter. Repeated letters stand for the same digit each time they appear.

$$\begin{array}{r} W\ E \\ \underline{M\ E} \\ S\ S\ S \end{array}$$

Find out what digits the letters stand for. (Hint: SSS must be a multiple of 111.)

5-3. Primes.

In this chapter we shall be interested in numbers as multiples of other numbers. A long time ago, in 200 B.C., a man named Eratosthenes thought of a process which separates all the numbers into two groups. In one group are all the numbers which are products of two smaller numbers. In the other group are numbers which are not products of two smaller numbers. This process is called the "sieve of Eratosthenes."

We shall make a sieve for the numbers from 1 to 100. On your paper arrange the numerals from 1 to 100 in 10 rows of 10 numerals each, as follows. Do not write in this book.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Mark a ring around the 2 on your paper. Then cross out all the other numerals which are multiples of 2.

Now start with the 3. Mark a ring around 3. Cross out all the rest of the numerals which are multiples of 3. You should have crossed out 6 and 9 in the top row. Go on through all the rows in the same way. Some numerals will be crossed out more than once.

The next numeral not crossed out is 5. Circle the 5 and cross out all the following multiples of 5. You may count by 5's to help you do this. Continue this process until all the numerals have been crossed out or circled.

The numerals that are circled are the names of the prime numbers that are less than 100.

Definition: *A prime number* is a counting number other than 1 that is divisible by itself and by 1 but not by any other counting number.

Exercises 5-3a

1. What did you find when you came to the multiples of 11?
2. When did you find that all the numerals you wished to cross out were already crossed out?

3. List the prime numbers less than 100.

4. How many prime numbers are less than 100?

5. How many prime numbers are less than 50?

6. What would have happened if we had started by crossing out the multiples of one?

7. List the first ten counting numbers which are multiples of 10.

8. List the set of numbers less than 50 that are multiples of 7.

9. List the set of numbers that are multiples of 3, and are greater than 30 but less than 60.

10. List the set of numbers which are less than 100 and are also multiples of both 3 and 5.

11. a. How many pairs of prime numbers less than 100 have a difference of 2?

    b. List these pairs. Such pairs are called twin primes.

12. Are there three numbers that might be called prime triplets?

13. a. List the numbers from 1 to 20.

    b. Underline the numerals in every second position, starting with 1.

    c. Circle the numerals for the prime numbers.

    d. Did you need to circle any numeral that was not underlined?

*14. What is the intersection of the set of prime numbers and the set of odd numbers less than 30?

*15. a. Why is it not necessary to use multiples of 8 in making your sieve?

b. Why was 7 the last prime number whose multiples were crossed out in making the sieve of prime numbers from 1 to 100?

*16. Do you think that there are more prime numbers between 100 and 200 than there are between 1 and 100? Explain your answer.

*17. Do you think that there will be some twin primes in the set of prime numbers between 100 and 200? Find one pair.

*18. Do you think that there will be some prime triplets in the set of prime numbers between 100 and 200?

The questions in the exercises above are the kinds of questions about which many people are curious. They study arrangements of numbers like the sieve you have made and look for patterns of numbers. You may like to look at your sieve to see what else it brings to your mind.

It is known that there is no largest prime number. No matter how large the number you find, there will be a larger number which is a prime. It is not difficult to show this.

Mathematicians have asked other questions about primes which have never been answered. Perhaps you may someday answer one and be world-famous for it.

These are some of the mysteries of prime numbers:

No one knows whether there is a number beyond which there are no more twin primes.

Mathematicians have searched for years for laws about the distribution of primes.

Experiment seems to show that even numbers after 3 can be expressed as the sum of two primes. No one has been able to prove that this is true for every even number.

Exercises 5-3b

1. Find a series of 7 consecutive numbers that does not contain a prime number. Use your sieve if you like.
2. Make a table like the following and complete the second column.

NUMBERS	TWIN PRIMES CONTAINED IN THIS SERIES
1-16	(3,5), (5,7), (11,13),
17-32	(17,19), (29,31)
33-48	
49-64	
65-80	
81-96	
97-112	
113-128	

   a. How many twin primes are there in each series?
   b. Do the twin primes seem to appear less often as the series become larger?
3. Why does the difference of 1 between primes occur only once?
4. Why is 2 the only even prime?
5. Find pairs of consecutive primes where the difference is 6.
6. What is the largest difference between any two consecutive primes which are less than 100 ?
7. Express each of the following even numbers as the sum of two primes.

   a. 12
   b. 14
   c. 16
   d. 50
   e. 76
   f. 94
8. **BRAINBUSTER.** Can you name the next number in the following sequence?

   11, 31, 41, 61, 71, 101, 131, 151, ...

5-4 Factors.

In Section 5-2 you worked with multiples. For example, since $3 \cdot 8 = 24$ we called 24 a multiple of 3 and of 8. A name is needed for the 3 and 8 which are multiplied to get 24. In the elementary grades you may have called them the multiplier and the multiplicand. Here we shall give them both the same name. The name which mathematicians use is factor.

Example 1: Write 12 as the product of factors.

$$12 = 1 \cdot 12$$
or $$12 = 2 \cdot 6$$
or $$12 = 3 \cdot 4$$
or $$12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$$

When we say "the factors" we mean "all the factors" of a number. For example, the number six has four factors: 1, 2, 3 and 6. The number 1 is a factor of every number. The number 1 has only one factor which is 1 itself. A number is always a factor of itself. The set of factors of any number besides 1 always includes two numbers, 1 and the number itself.

Example 2: Write the set of factors of 20.

The set of factors of 20 is $\{1, 2, 4, 5, 10, 20\}$

As you can see, factors are connected with multiplication. The number 2 is a factor of 28 if we can find a counting number which we can multiply by 2 to get 28. If we divide 28 by 2, we get 14. Then

$$2 \cdot 14 = 28$$

and 2 is a factor of 28 and so is 14.

Example 3: Write 6 as the product of two factors.

$$\underset{\text{product}}{6} = \underset{\text{factor}}{3} \cdot \underset{\text{factor}}{2} \quad \text{or} \quad \underset{\text{product}}{6} = \underset{\text{factor}}{6} \cdot \underset{\text{factor}}{1}$$

Notice that 6 is a multiple of 3, and 3 is a factor of 6. Also, 6 is a multiple of 2, and 2 is a factor of 6. The second expression $6 = 6 \cdot 1$ is important, too. It is the easy case which you can always think of.

A good way to make a statement about the factors of any number is to use symbols. We can use letters to stand for the product and the factors. This lets us talk about all numbers and not just 28 or 6 as we did above. This is how mathematicians define factor.

We have written

$$2 \cdot 14 = 28$$
$$3 \cdot 2 = 6$$
$$\text{factor} \cdot \text{factor} = \text{product}$$

Suppose we let a stand for 2, c stand for 14, and b stand for 28. Then we may write

$$a \cdot c = b \quad \text{for} \quad 2 \cdot 14 = 28$$

Next suppose we let a stand for 3, c stand for 2, and b stand for 6. We may write

$$a \cdot c = b \quad \text{for} \quad 3 \cdot 2 = 6.$$

We may write more examples such as

$$5 \cdot 9 = 45$$
$$11 \cdot 26 = 286$$
$$8 \cdot 8 = 64$$
$$1 \cdot 7 = 7$$

But we can never write them all. Let us think that a and c stand for any whole numbers we wish, and that b stands for their product. Then we write:

$$a \cdot c = b$$

to include all of the above statements and also any other statement like them.

Definition: Whenever we have three whole numbers a, b, and c where $a \cdot c = b$, we say that a is a factor of b and c is a factor of b.

Exercises 5-4a

1. Write each of the following as the product of two factors. Avoid using 1 as a factor whenever you can.
    - a. 15
    - b. 14
    - c. 33
    - d. 32
    - e. 42
    - f. 54
    - g. 13
    - h. 1
    - i. 0
2. List the set of all factors of each of the following:
    - a. 10
    - b. 31
    - c. 18
    - d. 21
    - e. 77
    - f. 30
    - g. 13
    - h. 9
3. List the numbers in Problem 2 that have only two factors.
4. What number multiplied by any other number always results in zero as the product?
5. A common factor of two or more whole numbers is a counting number which is a factor of each of the given numbers. Use your answers to Problem 2 to answer the following:
    - a. What are four common factors of 10 and 30?
    - b. What is the largest common factor of 9 and 21?
    - c. What is the only common factor of 9 and 10?
    - d. What are the common factors (other than 1) of 18 and 30?
6. Replace the "?" by a whole number so that the statement will be correct:
    - a. $7 \cdot ? = 35$
    - b. $? \cdot 8 = 64$
    - c. $2 \cdot 3 \cdot 5 = ?$
    - d. $0 \cdot 11 = ?$
    - e. $19 \cdot ? = 19$
    - f. $0 = 0 \cdot ?$
    - g. $3 \cdot 7 \cdot ? = 42$
    - h. $100 = 5 \cdot 5 \cdot 2 \cdot ?$
    - i. $5 \cdot ? = 65$
    - j. $51 = ? \cdot 3$
    - k. $3^2 \cdot 2 = ?$
    - l. $24 = 2 \cdot 2 \cdot ? \cdot 3$

When we factored 12 we had $12 = 2 \cdot 2 \cdot 3$. Something special is true of these factors. Do you know what it is? They are all prime numbers. When we factor a number into prime factors, we say that we have the complete factorization of the number.

Example: We write the complete factorization of the numbers 10, 8, 40 and 17 as follows:

$$\begin{aligned} 10 &= 2 \cdot 5 \\ 8 &= 2 \cdot 2 \cdot 2 = 2^3 \\ 40 &= 2 \cdot 2 \cdot 2 \cdot 5 = 2^3 \cdot 5 \\ 17 &= 17 \end{aligned}$$

Notice that 1 does not appear in the complete factorization because it is not a prime number. To factor a number completely, or to find the prime factors of a number, you may wish to proceed as in the following example.

Example: a. $8 = 4 \cdot 2$

$$8 = \overbrace{2 \cdot 2} \cdot 2 = 2^3$$

b. $100 = 10 \cdot 10$

$$100 = \overbrace{2 \cdot 5} \cdot \overbrace{2 \cdot 5}$$

$$100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 \cdot 5^2$$

Observe that $8 = 4 \cdot 2$ shows factors of 8 but not the complete factorization. Also $100 = 5 \cdot 2 \cdot 10$ shows factors of 100 but not the complete factorization. Explain why these factorizations are not complete.

Oral Exercises 5-4b

1. Tell which of the following have more than just two different factors:

   Example: 10 has factors 1, 2, 5 and 10.
   4 has factors 1, 2, 4
   29 has factors 1, 29
   10 and 4 have more than two different factors. 29 has only two different factors.

   a. 2
   b. 9
   c. 26
   d. 61
   e. 133
   f. 97
   g. 52
   h. 79

2. Wherever possible, express each of the above numbers as the product of two smaller numbers.

3. Give the complete factorization of the following:

   a. 6 b. 15 c. 16 d. 12 e. 31 f. 35

Numbers like those in Problem 2 above are called composite numbers.

Definition: A composite number is a number which can be expressed as a product of two smaller whole numbers.

Naturally, it may be possible to factor further so that there are more than two smaller numbers.

Example:

$$16 = 2 \cdot 8$$
$$16 = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4$$

and

$$98 = 2 \cdot 49$$
$$98 = 2 \cdot 7 \cdot 7 = 2 \cdot 7^2$$

The numbers you crossed out when you made your sieve in Section 5 are all composite numbers. Explain why they are composite numbers.

The number six is a composite number since $2 \cdot 3 = 6$. The only way the number 13 can be factored is $1 \cdot 13 = 13$. The factors are not both smaller than 13. Therefore 13 is a prime number, not a composite number.

Exercises 5-4c

1. Tell which factorizations are complete and which are not.

   a. $25 = 5 \cdot 5$
   b. $102 = 6 \cdot 17$
   c. $81 = 27 \cdot 3$
   d. $22 = 2 \cdot 11$
   e. $138 = 3 \cdot 2 \cdot 23$
   f. $12 = 4 \cdot 3$

2. Tell whether each of the following numbers is prime or composite.

   a. 10 b. 31 c. 9 d. 18 e. 27 f. 24 g. 11

3. Write the complete factorization of the composite numbers in Problem 2.

4. Factor the numbers listed in as many ways as possible, using only two factors at a time. Because of the commutative property, we know that $5 \cdot 3$ is not different from $3 \cdot 5$.

   a. 26
   b. 38
   c. 36
   d. 68
   e. 81
   f. 100

5. a. According to our definition of factor, is zero a factor of 6?

   b. Is 6 a factor of zero? Explain your answers.

6. a. Write the set of factors of 20.

   b. Write the complete factorization of 20.

   c. What factors of 20 do not appear in the complete factorization of 20?

7. Write the complete factorization of:

   a. 75
   b. 64
   c. 105
   d. 45
   e. 56
   f. 50

Let us factor 42 in as many ways as possible using two factors at a time, and omitting $42 \cdot 1$. The various possibilities are:

a. $2 \cdot 21$
b. $3 \cdot 14$
c. $6 \cdot 7$

Now if we factor again the factors in each pair until we have prime factors, we have:

a. $2 \cdot 3 \cdot 7$
b. $3 \cdot 2 \cdot 7$
c. $2 \cdot 3 \cdot 7$

What conclusions do you reach about the results? The property of counting numbers you are asked to notice is very important and has been given a name. It is called the Unique Factorization Property of the Counting Numbers. It means that every counting number can be factored into primes in only one way except for order.

Example: $20 = 5 \cdot 2 \cdot 2$ $20 = 2 \cdot 5 \cdot 2$ $20 = 2 \cdot 2 \cdot 5$

These three examples show the same prime factors of 20. Since only the order is different, we say there is really only one complete factorization.

5-5. Divisibility.

In mathematics as well as in English we can say the same thing in several ways.

1. $24 = 8 \cdot$ (some whole number)
2. 24 is a multiple of 8
3. 8 is a factor of 24
4. 24 is divisible by 8

These four statements say the same thing in different ways. We have been using the first three ways and are ready to use the fourth. To say that 24 is divisible by 8 means that when we divide 24 by 8, we get a whole number.

All four statements may be collected into one if we use the letter a to stand for the whole number in 1. We write

$$24 = 8 \cdot a$$

This single statement tells that 24 is a multiple of 8 and also that if 24 is divided by 8, the answer is the whole number, a.

Then $24 = 8 \cdot a$

also means $24 \div 8 = a$.

You will use this idea in the next chapter.

Our problem now is to find out whether a number is divisible by a number other than itself or 1. We can simply guess and try, but we do not have to. We can use what we know about numbers and find short-cuts, at least for some special divisors.

From the sieve of Eratosthenes which you made, you learned that numerals which have 0, 2, 4, 6, 8 in the ones place represent numbers which are divisible by 2. These numbers are called even numbers.

Definition: A whole number which is divisible by two is an even number.

A whole number which is not divisible by two is an odd number.

Exercises 5-5a

1. Is it possible for a whole number to be neither even nor odd?
2. Is zero even or odd? Why?
3. Tell whether these expressions represent even or odd numbers:
    - a. $2 \cdot 5$
    - b. $3 + 7$
    - c. $6 \cdot 5 \cdot 3$
    - d. $2 + 16$
    - e. $7 + 8$
    - f. $3 \cdot 2 \cdot 9$
    - g. $128 - 37$
    - h. $26 \div 2$
4. Complete the statements by filling the blank with either "odd" or "even".
    - a. The sum of two even numbers is always _____.
    - b. The sum of two odd numbers is always ______.
    - c. The sum of an odd and even number is always ______.
    - d. Zero is an _______ number.
    - e. The product of two even numbers is always ________.

f. The product of two odd numbers is always ______.

g. The product of an even and odd number is always ______.

h. The sum of three odd numbers is always ______.

i. The sum of seven odd numbers is always ______.

j. The sum of thirty-five odd numbers is always ______.

k. The sum of four odd numbers is always ______.

l. The sum of two odd numbers and three even numbers is always ______.

5. Tell whether these numbers are even or odd. Must you divide or can you tell by looking at the numerals?

a. 50
b. 97
c. 1006
d. 999
e. 1,406,700
f. 7,531,918

6. The numerals above were written in the decimal system. Try to classify numbers as even or odd when their numerals are written in a different system.

a. XIV b. 12_{five} c. 11_{three} d. CCC

7. BRAINBUSTER: Is divisibility a property of a numeral or a property of a number? Explain your answer.

8. BRAINBUSTER: Is the number $2 \cdot a$, where a is a whole number, even or odd? Why?

In solving problems about factors we often use the term divisible. We remember that

$2 \cdot 3 = 6$ also means $6 \div 2 = 3$ or $6 \div 3 = 2$

and that $35 \div 5 = 7$ also means $5 \cdot 7 = 35$.

In the same way $72 \div 8 = 9$ also means $8 \cdot 9 = 72$

and finally, $b \div a = c$ also means $a \cdot c = b$, if a is not zero. Why do we say that a is not zero? We say that the whole number b is divisible by the counting number a if there is a whole number c so that $a \cdot c = b$.

This tells us that b is divisible by a if a is not zero and b is the product of a and a whole number. Then to say that b is divisible by a is the same as saying that b is a multiple of a.

From the exercises above, we find that in our way of writing the names of numbers, we can look at a numeral and tell if it is divisible by 2. By looking at b we can decide if there is a whole number c so that

$$2 \cdot c = b.$$

Let us look for a short-cut way to tell whether a number is divisible by 3. To do this we make a table. In the first row we list some numbers. In the second row place an x if the number is divisible by 3 and an 0 if it is not. In the third row write the sum of the digits.

Number	21	111	13	142	51	45	53	48	57	96	477	209
Divisible by 3	x	x	0									
Sum of digits	3											

What do you notice about the sum of the digits which seems to be true just for the numbers which are divisible by 3?

Can you guess a general rule for telling whether a number is divisible by 3? Does your guess work for 87? For 166?

Exercises 5-5b

1. Test the following numbers for divisibility by 3. Write yes or no.

   a. 51
   b. 92
   c. 86
   d. 213
   e. 10,002
   f. 1,452

2. Divide each number in Problem 1 by 3. Show the quotient and the remainder if there is a remainder.

3. Write the multiples of 10 from 10 to 100. Look for the pattern. State a test for divisibility by 10. Check your test.

4. Write the multiples of 5 from 5 to 80. Look for the pattern in this set of numerals and state a test for divisibility by 5. Give examples.

5. This is a test for divisibility by 9: A number is divisible by 9 if the sum of the digits of its numeral is divisible by 9. Give some examples for which this is true.

6. This is a test for divisibility by 4: A whole number is divisible by 4 if the numerals in the last two places on the right name a number which is divisible by 4. Try 128, 5012, 413, and 109, and other numbers which you think of.

7. Use the divisibility tests above and circle the factors of the numbers below. Copy the numerals. Do not write in your book.

Examples:	360	②	③	④	⑤	⑨	⑩
	225	2	③	4	⑤	⑨	10
a.	124	2	3	4	5	9	10
b.	210	2	3	4	5	9	10
c.	136	2	3	4	5	9	10
d.	615	2	3	4	5	9	10
e.	207	2	3	4	5	9	10
f.	5316	2	3	4	5	9	10
g.	10,008	2	3	4	5	9	10
h.	4041	2	3	4	5	9	10
i.	30,420	2	3	4	5	9	10
j.	7776	2	3	4	5	9	10
k.	1961	2	3	4	5	9	10
l.	1490	2	3	4	5	9	10

8. If a number is divisible by 9, is it also divisible by 3?

9. a. If a number is divisible by 2 and by 3, is it also divisible by 6?
   b. If a number is divisible by 6, is it also divisible by 2 and 3?

10. Which of the following are divisible by 6?
    a. 144 b. 102 c. 748 d. 504

Tests for Divisibility

A number is divisible by	if this condition is met:
2	The numeral in the ones place is 0, 2, 4, 6, 8.
3	The sum of the digits is a multiple of 3.
4	The last 2 digits on the right name a multiple of 4.
5	The digit in the ones place is 0 or 5.
6	The numeral meets the conditions for both 2 and 3, above.
9	The sum of the digits is a multiple of 9.
10	The digit in the ones place is 0.

All of these tests which use the way a numeral looks are useful in the case of decimal numerals. The number twenty-one is still divisible by 3 no matter how we write it. Divisibility is a property of a number. The tests we use in our decimal system are properties of the numerals. You can use our test for divisibility by 3 when twenty- one is written 21. When twenty-one is written XXI our test does not apply.

Let us use the tests we know in some problems.

Example: Find the complete factorization of 90.

1. Since the last digit is 0, the number is even and has 2 as a factor. $90 = 2 \cdot 45$
2. Since the last digit in 45 is 5, the number is divisible by 5. $90 = 2 \cdot 5 \cdot 9$
3. Nine is divisible by 3.
   $90 = 2 \cdot 5 \cdot 3 \cdot 3$ or $90 = 2 \cdot 3 \cdot 3 \cdot 5$ or
   $90 = 2 \cdot 3^2 \cdot 5$.

Example: Find the complete factorization of 495.

1. The number is not even since the last digit is not even. Then 2 is not a factor.
2. The last digit is 5 so five is a factor.

$$495 = 5 \cdot 99$$

3. 99 is divisible by 9 at sight.

$$495 = 5 \cdot 9 \cdot 11$$

$$495 = 3 \cdot 3 \cdot 5 \cdot 11 = 3^2 \cdot 5 \cdot 11$$

Example: Find the complete factorization of 249.

1. The number is not even so we shall not try 2.
2. The sum of digits is 15 so 249 is divisible by 3.

$$249 = 3 \cdot 83$$

3. The sum of the digits of 83 is 11 so 83 is not divisible by 3.
4. The numeral 83 does not end in 5 or 0 so the number is not divisible by 5.
5. We try 7 and 11 and find that 83 is prime. The complete factorization is $249 = 3 \cdot 83$.

Exercises 5-5c

What is the smallest prime factor of each of the following?

1. 115
2. 162
3. 77
4. 110
5. 129
6. 1466
7. 1125
8. 143

Find the complete factorization of the following:

9. 46
10. 65
11. 114
12. 98
13. 180
14. 258
15. 486
16. 444
17. 375
18. 539
19. 9
20. 100
21. 64
22. 37
23. 100,000
24. 11
25. 1000
26. 32

*27. In Problems 19-26 on the previous page, find the complete factorization of the squares of each number.

*28. How many times does each factor of the number in Problems 19-26 appear in the factorization of the square of the number. Can any factor appear an odd number of times in any number which is the square of a whole number?

*29. What numbers have exactly 3 different factors?

*30. Write all the factors of 6 smaller than 6 itself. Find the sum of these factors. Write all the factors of 28 smaller than 28 itself. Find the sum of these factors. Write all the factors of two other numbers less than 30. Find the sum of the factors of each number. What do you notice in the case of 6 and 28? Such numbers were called "perfect numbers" by the Greeks. No one knows how many perfect numbers there are or whether there are any odd perfect numbers. If you find an odd number which is perfect, run, don't walk, to the nearest mathematician.

5-6. Least Common Multiple.

You have already learned a great deal about multiples of numbers:

that all whole numbers are multiples of 1,

that even numbers {0, 2, 4, 6, 8, 10, 12, ...} are multiples of 2,

that {0, 3, 6, 9, 12, ...} are multiples of 3.

Similarly, we can list the multiples of any counting number.

The number 2 is an even number, and the number 3 is an odd number. Usually we do not think of 2 and 3 as having much in common. Yet if we look at the sets of multiples for 2 and 3, we see they do have something in common. Some of the multiples of 2 are also multiples of 3. For example, 6 is a multiple of both 2 and 3. There are many such numbers divisible by both 2 and 3. The set of these numbers is written as follows:

{6, 12, 18, 24, 30, ...}

Definition: A number which is a multiple of more than one number is called a common multiple of these numbers.

Thus, 12 is a common multiple of 2 and 3.

Let's try another example. List the common multiples of 3 and 4. First, list the multiples of each:

Set of multiples of 3: {0, 3, 6, 9, 12, 15, 18, 21, 24, ...}

Set of multiples of 4: {0, 4, 8, 12, 16, 20, 24,...}

The numbers that these sets have in common are the common multiples of 3 and 4. This set is written as follows:

{0, 12, 24, 36, 48,...}

Suppose we wish to find the least common multiple of 12 and 18. First, we list the sets of multiples of each:

Set of multiples of 12: {0, 12, 24, 36, 48, 60, 72, 84,...}

Set of multiples of 18: {0, 18, 36, 54, 72,...}

The set of common multiples of 12 and 18 is {0, 36, 72, 108,...}.

We call the smallest counting number in this set the least common multiple. The smallest counting number in this set is 36. Zero is not a counting number. Therefore, we call 36 the least common multiple of 12 and 18.

Definition: The least common multiple of a set of counting numbers is the smallest counting number which is a multiple of each member of the set of given numbers.

What is the least common multiple of 2, 3, and 4?

Set of multiples of 2: {0, 2, 4, 6, 8, 10, 12,...}
Set of multiples of 3: {0, 3, 6, 9, 12, 15,...}
Set of multiples of 4: {0, 4, 8, 12, 16, 20,...}

The set of common multiples of 2, 3, and 4 is:
{0, 12, 24, 36,...}

What is the smallest counting number in this set? According to our definition, the least common multiple of 2, 3, and 4 is 12.

The least common multiple is useful in work with fractions. You may remember that you often needed the least common denominator in addition and subtraction of fractions. The least common denominator of several denominators is their least common multiple. You really do not need the name, least common denominator.

Consider the example:

$$\frac{3}{4} + \frac{5}{6} + \frac{1}{12}$$

The denominators are 4, 6, and 12. We wish to find the least common multiple of these three numbers.

Set of multiples of 4: {0, 4, 8, 12, 16, 20, 24,...}

Set of multiples of 6: {0, 6, 12, 18, 24,...}

Set of multiples of 12: {0, 12, 24,...}

Set of common multiples: {0, 12, 24,...}

The least common multiple of 4, 6, and 12 is 12.

Try another example:

$$\frac{5}{6} - \frac{8}{15}$$

Set of multiples of 6: {0, 6, 12, 18, 24, 30, 36,...}

Set of multiples of 15: {0, 15, 30,...}

The least common multiple is 30. Sometimes you can see the least common multiple without writing out the set of common multiples.

Note that 0 is a common multiple for any set of whole numbers. It is not a common multiple for a set of counting numbers. In adding or subtracting fractions, 0 cannot be used as a common denominator. Can you write $\frac{1}{2}$ with a zero denominator?

Exercises 5-6

1. Write the set of all multiples less than 100 for each of the following.

   a. 6 b. 8 c. 9 d. 12

   Use this form:

   a. The set of multiples of 6 less than 100: {? ......}.

2. Using your answers in Problem 1, write the set of all common multiples less than 100 for each of the following sets.

   a. 6 and 8 b. 6 and 12 c. 8 and 12 d. 8 and 9

   Use this form:

   a. The set of common multiples of 6 and 8 less than 100: {? ....}.

3. Using your answers in Problem 2, write the least common multiple for each of the sets in Problem 2.

   Use this form:

   a. The least common multiple of 6 and 8 is __?__ .

4. Find the least common multiple for each of the following sets:

   a. 2, 5
   b. 4, 6
   c. 2, 3, 5
   d. 3, 4, 6
   e. 2, 5, 6
   f. 4, 5, 6
   g. 2, 6, 7
   h. 8, 9, 12

5. Find the least common multiple for each of the following sets.

   a. 2, 3
   b. 3, 5
   c. 3, 7
   d. 5, 7
   e. 2, 11
   f. 2, 13
   g. 7, 11
   h. 3, 13
   i. 11, 13
   j. 2, 3, 5

6. Give a definition of the least common multiple of 3 numbers.

7. Give a definition of the least common multiple of any set of numbers.

8. a. What is the least common multiple of 1 and 6?

   b. Of 6 and 6?

9. How does the size of the least common multiple of a set of numbers compare with the numbers themselves?

10. Find the least common multiple for each of the following sets.
    - a. 4, 6
    - b. 4, 8
    - c. 4, 10
    - d. 6, 9
    - e. 8, 10
    - f. 10, 12
    - g. 12, 15
    - h. 4, 6, 10
    - i. 10, 15, 30
    - j. 4, 6, 8

*11. Study the following examples. Try to discover a shorter way to determine the least common multiple.

Example A. To find the least common multiple of 4, 6, and 8:

1. First, write the complete factorization for each number.
   $4 = 2^2$ $6 = 2 \cdot 3$ $8 = 2^3$
2. The least common multiple is $2^3 \cdot 3$ or 24.
3. Note that $2^2 \cdot 2 \cdot 3 \cdot 2^3 = 192$ which is a common multiple of 4, 6, and 8, but not the least common multiple.

Example B. To find the least common multiple of 12 and 18:

1. The complete factorization for each number is:
   $12 = 2^2 \cdot 3$ $18 = 2 \cdot 3^2$
2. The least common multiple of 12 and 18 is
   $2^2 \cdot 3^2$ or 36.
3. Is $(2^2 \cdot 3 \cdot 2 \cdot 3^2)$ a common multiple of 12 and 18?

Now find the least common multiple for each of the following sets.

- a. 12, 16
- b. 16, 18
- c. 9, 15
- d. 10, 14
- e. 14, 16
- f. 4, 5, 6
- g. 6, 8, 9
- h. 8, 9, 10
- i. 12, 20, 22
- j. 9, 16, 20

*12. a. Is there a greatest common multiple of 3 and 5? If so, write an example.

b. Is there a greatest common multiple of 4 and 6? If so, write an example.

c. Is there a greatest common multiple of any set of counting numbers?

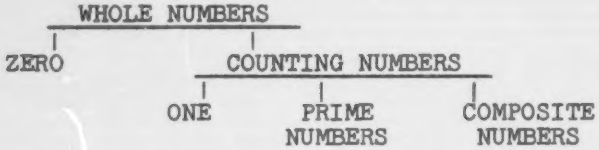
*13. a. Observe that the product of several counting numbers is a common multiple of each of the factors. Under what condition is the product of two counting numbers the least common multiple of the two counting numbers?

b. When is the product of three counting numbers the least common multiple of the three counting numbers?

*14. A man who had 20 lumps of sugar said he put them all in 3 cups of coffee so that he had an odd number of lumps of sugar in each cup. How could he do this?

5-7. Summary.

This chapter deals entirely with whole numbers. A way to think of subsets of whole numbers is shown in this sketch.



Note that ZERO is a member of the set of whole numbers but not of the set of counting numbers. ONE, the PRIME NUMBERS, and the COMPOSITE NUMBERS are members of the set of COUNTING NUMBERS and also the set of WHOLE NUMBERS. The set of counting numbers is a subset of the set of whole numbers.

Counting numbers, other than one, are either prime or composite. A composite number has factors different from 1 and the composite number itself. A prime number has exactly two different factors, 1 and the prime number itself. One is not a prime number.

A composite number can be factored into prime factors. No matter what steps are used in arriving at the prime factors of a given composite number, they will be the same, except for order.

Certain tests for divisibility are available to help in finding the factors of a composite number. Inspection of the numeral will give information about divisibility of the number by 2, 3, 4, 5, 6, 9, and 10.

The least common multiple (L.C.M.) of several numbers is often useful in computation with fractions. The L.C.M. of several given numbers may be found by using sets of multiples or by using prime factors of the given numbers.

Some important terms used in this chapter are:

1. MULTIPLE. The whole number, b, is a multiple of the whole number, a, if $a \cdot c = b$ where the number, c, is a counting number.
2. A PRIME number is a counting number, other than 1, which is divisible by itself and by 1 but not by any other counting number. The number 1 is not a prime number.
3. A COMPOSITE number is a counting number, other than 1, which is not prime. Composite numbers have more than two factors.
4. FACTOR. The number, a, is a factor of b if $a \cdot c = b$, where a, b, and c are whole numbers. The term factor is used instead of the words "multiplier" and "multiplicand". The number a is a factor of the number b if b is divisible by a.

5. A COMMON FACTOR of two or more whole numbers is a counting number that is a factor of each of the given numbers.
6. The COMPLETE FACTORIZATION of a counting number represents that number as a product of prime numbers. The complete factorization of a prime number is the number itself. Complete factorization is useful in finding the least common multiple of two or more numbers.
7. The LEAST COMMON MULTIPLE of two or more counting numbers is the smallest counting number which is a multiple of each of the given numbers. The least common multiple of a set of numbers cannot be less than the largest member of the set of numbers.

5-8. Chapter Review.

Exercises 5-8

1. Name the five smallest prime numbers.
2. What is the intersection of the set of prime numbers and the set of odd numbers greater than 40 but less than 60?
3. Is there an even prime number? If so, what is it?
4. Express as the sum of 2 primes.
    a. 8
    b. 24
5. Name a pair of primes for which the difference is 1. Are there any others? Why?
6. Write the set which consists of the first five common multiples of 2 and 5.
7. Using exponents, write the prime factors of 144.
8. Label the following as composite or prime.
    a. 70
    b. 71
    c. 72
    d. 73
    e. 74
    f. 75
9. Write the set of factors for each of the following.
    a. 8
    b. 12
    c. 16
    d. 19
    e. 21
    f. 28
10. Using your answers to Problem 9, write the set of all common factors in each of the following cases.
    a. 8 and 12
    b. 12 and 21
    c. 16 and 19
    d. 16 and 28
    e. 8, 12, and 16
    f. 16, 19, 21, and 28
11. Factor each number completely.
    a. 35
    b. 4
    c. 27
    d. 66
    e. 60
    f. 147
12. Write the complete factorization of each number.
    a. 78
    b. 56
    c. 18
    d. 250
    e. 96
    f. 110
13. a. Is there a least common factor of 6 and 9? of any two counting numbers?
    b. Is there a greatest common factor of 6 and 9?

14. Find the least common multiple of each set of numbers.
    a. 20, 25
    b. 14, 21
    c. 4, 11
15. If c is a whole number and $2 \cdot c = b$, what is known about b?
16. Use the divisibility tests in this chapter and circle the factors of the numbers below. Copy the numerals. Do not write in your book.

a.	165	2	3	4	5	6	9	10
b.	438	2	3	4	5	6	9	10
c.	120	2	3	4	5	6	9	10
d.	145	2	3	4	5	6	9	10
e.	639	2	3	4	5	6	9	10

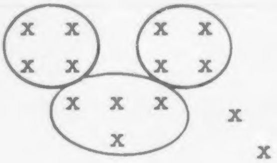
17. Find the least common multiple of each set.
    a. 3, 9, 12
    b. 2, 3, 5
    c. 2, 8, 16
    d. 4, 5, 8
    e. 4, 6, 8
    f. 8, 10, 12

18. Find the greatest common factor of each set.
    a. 6, 8
    b. 7, 14
    c. 22, 55
    d. 60, 108, 24
    e. 194, 166
    f. 535, 305
19. Do you think there is a largest prime number? Can you find it, or can you give a reason for thinking as you do?
20. If a number is divisible by 4, is it (always, sometimes, never) divisible by 2? Choose the word which gives the best answer.
21. If a number is divisible by 2, is it (always, sometimes, never) divisible by 4? Choose the word which gives the best answer.
22. One bell strikes every three minutes and another bell strikes every five minutes. If the bells strike together to begin with, how long will it be before they strike together again?

5-9. Cumulative Review.

Exercises 5-9

1. How many symbols are needed for a place value numeration system with base 20?
2. The x's below are grouped so that the number of x's can easily be written in a numeral to some base.
   a. What is the base? b. Write the numeral in that base.



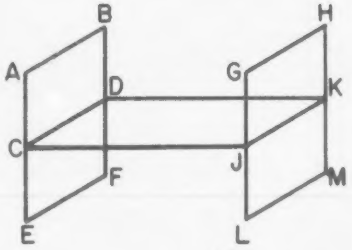
3. Which of the following numerals represents the largest number?
   a. 28_{ten} b. 23_{twelve} c. 10000_{two} d. 37_{eight}
4. Write the decimal numeral for:
   a. 10101_{two} b. 41_{five} c. 27_{twelve}
5. If N represents an even number, then $N + 1$ will be
   a. even b. odd
   c. we can't be sure until we know what N is.
6. Write 2,304 in expanded form.
7. a. The inverse operation of multiplication is __________.
   b. The inverse operation of addition is ____________.
8. Are the following sets closed under addition?
   a. $\{2, 4, 6, 8, 10, \ldots\}$
   b. $\{7, 14, 21, 28, 35\}$
9. Find a whole number or numbers which can be used for x to make the following true.
   a. $7 + x = 15$
   b. $1 + x = x + 1$
   c. $x = 0 + 3$
10. State the associative property of multiplication in words and symbols.

11. Write the following statements in words.

    a. $8 < 12$ b. $34 > 32$ c. $5 > 3 > 2$

12. In the figure on the right, name:

    a. plane AFE $\cap$ plane CJK
    b. plane GLM $\cap$ plane ABF
    c. two skew lines
    d. $\overleftrightarrow{BF} \cap \overleftrightarrow{CD}$
    e. $\overleftrightarrow{JC} \cap$ plane BFE



INDEX

addends, 42
addition
  in base five, 42-46
  of decimals, 315-316, 339
  of rational numbers, 221-222, 224, 226
  on the number line, 270, 300-301
angle, 247-249, 250-251, 263
  definition, 247, 263
  exterior of, 248
  interior of, 248
  notation, 248
  of a triangle, 250
  sides, 247
  vertex, 247
approximately equal to, 371
associative property, 81-86, 91, 110
  of addition, 82-83, 85, 86, 91, 110
  of multiplication, 83-84, 85, 86, 91, 110
Babylonian numerals, 19
base, 23, 31, 65, 66
base changing, 55-56
base five, 34-56, 66
  addition, 42-46
  division, 53
  multiplication, 49-52
  subtraction, 47-48
base seven, 57, 58
base ten, 28, 34, 35, 36, 64, 65, 66
base twelve, 59-60, 65
base two, 60-61, 65
betweenness, 100-101, 110, 235
binary system, 60-61, 65
boundary, 242, 260
budgets, 375-376
closed region, 260
closure, 92-93, 106, 110, 216, 218
commission, 378-380
common factor, 162, 179
common multiple, 174
commutative property, 75-81, 91, 110
  of addition, 80, 85, 91, 110
  of multiplication, 80, 85, 91, 110
Comparison Property, 281, 302, 321, 356, 391
complete factorization, 162-163, 179
complex fractions, 294-296, 302
composite numbers, 164-165, 178, 179
counting numbers, 73-74, 92, 102, 105, 106, 107, 108, 109, 153, 166, 178
curve, 258
  broken-line, 258
  simple closed, 258

decimals
   addition of, 315-316, 339
   division of, 321-323, 339
   expanded form, 312, 339
   expressed as a percent, 370
   multiplication of, 319, 339
   notation, 309, 312, 339
   numerals, 21
   places, 309
   point, 309
   rational numbers as decimals, 325-332
   repeating, 326-332, 340
   rounding, 335-337, 340
   subtraction of, 317, 339
   system, 21, 22, 23, 65
denominator, 208, 278, 280
digits, 22, 65
discounts, 378-380
distributive property, 86-90, 91, 110
divisibility, 166-172, 178
   by two, 167, 171
   by three, 169, 171
   by four, 170, ex.6, 171
   by five, 169, ex. 4, 171
   by nine, 170, ex. 5, 171
   by ten, 169, ex. 3, 171
   tests for, 171-172, 178
divisible, 166, 168
division
   in base five, 53
   meaning of, 192-194
   of decimals, 321-323, 339
   of rational numbers, 192-194, 216-218
   on the number line, 272-273
duodecimal system, 59-60, 65
Egyptian numerals, 17-18
elements of a set, 70, 110, 134, 250
empty set, 134, 147
endpoint, 236
equivalent fractions, 211-212, 297-298
Eratosthenes, sieve of, 155
Euclid, 113
Euclidean geometry, 113
even number, 75, 167
expanded form, 27-29, 65
   in base five, 40
   in decimals, 312
exponent, 31-32, 65 66
exterior
   of angle, 248
   of simple closed curve, 259
   of triangle, 251

factor, 31, 66, 160-166, 178, 179
  common factor, 162, 179
fractions, 185, 189-190, 302
  comparison of, 278, 280, 281-282
  complex, 294-296, 302
  definition, 189
  equivalent, 211-212, 297-298
  expressed as decimals, 371, 391
  expressed as mixed numbers, 287-288
  expressed as percents, 370, 391
  improper, 284, 302
  proper, 284, 302
Gauss, Karl Friedrich, 5
geometric line, 116, 126
geometry, 113, 144
greater than, 78, 101, 110
half-line, 244, 263
half-plane, 243
half-space, 242
Hindu-Arabic system, 21
identity element
  for addition, 106, 110
  for multiplication, 103, 110
if-then statements, 1-2
improper fractions, 284, 302
  expressed as a mixed number, 287-288
interest, 388-390
interior
  of angle, 248
  of simple closed curve, 259
  of triangle, 251
intersect, 117
intersections, 117, 133-141, 145-146, 147
  lines and planes, 140, 146, 147
  sets, 133-134, 145, 147
  two lines, 117, 137-139, 146, 147
  two planes, 135-136, 140-141, 146, 147
inverse operations, 95-98, 110
Königsberg Bridge problem, 13-14
least common multiple, 173-175, 179, 224
less than, 78, 101, 110
line, 116-118, 126, 132, 145, 244
  separation, 244
lines and space, 116-119
logical reasoning, 2
marked price, 379
mixed numbers, 286, 302
  addition of, 289-290
  division of, 293
  expressed as improper fractions, 289
  multiplication of, 293
  subtraction of, 290-291
multiple, 154, 173-175, 179

multiplicand, 160, 179
multiplication
  in base five, 49-52
  of decimals, 319, 339
  of rational numbers, 203-210
  on the number line, 272
multiplication property of 0, 200
multiplication property of 1, 322
multiplier, 160, 179
naming lines, 118, 126, 132, 145
naming planes, 127-129, 145
naming points, 114, 145
net price, 379
non-metric geometry, 113, 144, 235
not equal to, 78, 110
number line, 100-101, 110, 269-283, 300-301
numerals, 17, 64, 66
numerator, 208
odd number, 167
one, the number, 74, 102-104, 108, 110, 153, 178, 179
one-to-one correspondence, 73, 253-254, 263
order
  of rational numbers, 278, 301
parallel lines, 138, 146, 147
parallel planes, 141
percent, 360-367, 370
  applications, 375-391
  expressed as a decimal, 370
  expressed as a fraction, 370
  of increase and decrease, 384-385
  used for comparison, 382-383
perfect numbers, 173
place value, 22, 23-24, 28, 39, 55, 66, 313
plane, 120-124, 127-129, 145, 243
  separation, 243
point, 113-114, 126, 145
point of intersection, 117
positional notation, 312
powers, 29, 32, 66
primes, 155-158, 178, 179
  prime factors, 162, 163, 178
  prime numbers, 156, 178, 179
  twin primes, 157, 158, 159
principal, 388
product of two rational numbers, 207-210
proper fractions, 284, 302
properties of one, 102-104, 108
properties of operations with numbers, 200
properties of zero, 105-108
proportion, 355-356
  definition, 355
quotient of two rational numbers, 218
rate, 350

rate of the commission, 379
rate of interest, 388
ratio, 188, 347-354
    definition, 347, 391
    expressed as a percent, decimal, fraction, 369-373
rational numbers, 185, 187, 189-190, 229-230, 269
    addition of, 221-222, 224, 226
    comparison of, 276, 278, 280, 281-282
    decimal form, 325-332
    definition, 188, 190, 195
    division of, 192-194, 216-218
    expressed as mixed numbers, 286
    multiplication of, 203-210
    on the number line, 274-275, 276-277
    operations with, 230
    order of, 278, 301
    product of two rational numbers, 207-210
    quotient of two rational numbers, 218
    subtraction of, 226-227
    sum of two rational numbers, 226
ray, 246, 262
    notation 246
reciprocals, 215, 296
region, 260, 263
    closed, 260
repeating decimals, 326-332, 340
Roman numerals, 19-20
rounding decimals, 335-337, 340
sale price, 379
segment, 235-236, 262
    notation, 236
separations, 241-244, 263
    of line, 244
    of plane, 243
    of space, 242
set, 69-71, 110, 113, 133-134, 147, 238, 239
    intersection of, 133-134, 145, 147
    subset, 239
    union of, 238, 250, 262
similar triangles, 359
simple closed curves, 257-260, 263
    exterior of, 259
    interior of, 259
skew lines, 138, 146, 147
space, 113, 117, 119, 242
    separation, 242
straight line, 116
subscripts, 132
subset, 239, 262
subtraction
    in base five, 47-48
    of decimals, 317, 339
    of rational numbers, 226-227
    on the number line, 270-271, 301

sum of two rational numbers, 226
symbols, 17, 66, 126-129
    > (greater than), 78, 101, 110
    < (less than), 78, 101, 110
    ≠ (not equal to), 78, 110
    ≈ (approximately equal to), 371
    · (times), 78-79, 110
    ↔ (line), 118, 145
    — (segment), 236
    △ (triangle), 250
    ∠ (angle), 248
    → (ray), 246
    ∩ (intersection), 134, 145-146, 147
    ∪ (union), 238
times, notation, 78-79, 110
triangle, 250-251, 263
    angles of, 250
    exterior of, 251
    interior of, 251
    notation, 250
    sides of, 250
    similar, 359
    vertices of, 250
twin primes, 157, 158, 159
union
    of sets, 238, 250, 262
    notation, 238
unique factorization property, 166
vertex
    of angle, 247
    of triangle, 250
whole numbers, 74, 97, 105, 110, 178
zero, 74, 105-108, 110, 175, 178